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OPTIMAL CONTROL OF HIGH SPEED TRAINS

by



Lloyd Edwin Peppard

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF SCIENCE

DEPARTMENT OF ELECTRICAL ENGINEERING

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled Optimal Control of High Speed Trains submitted by Lloyd E. Peppard in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

In recent years the problem of providing high speed ground transportation for large urban populations has received considerable attention. In this thesis, a review of the basic concepts of high speed train operation is presented leading up to the development of a mathematical model for a system of trains. The model is based on a previous derivation using Newton's Second Law but is modified to include jerk (rate of change of acceleration) as a variable.

A train regulator is derived from known theory of optimal control (the linear state regulator problem with infinite terminal time) and the resultant control system simulated on an analogue computer. System behaviour is compared with that of the previously developed system using several performance indices in the form of cost functionals.

The optimal regulator developed in this thesis appears to better meet the objective that excessive jerk be minimized.

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NOMENCLATURE

<u>Symbol</u>	<u>Meaning</u>	<u>Symbol First Appears on Page</u>
<u>A</u>	constant nxn coeff. matrix of state equation	16
a	acceleration (or deceleration)	9
a_i	drag coeff. for ith train of 2nd-order model	17
a_M	max. deceleration during braking	11
a_0	fictitious constant emergency deceleration rate during braking	11
α	no. of trains on track 1 of merging operation	67
<u>B</u>	constant nxm coeff. matrix of state equation	16
b_i	constant of proportionality between $f_i(t)$ and $u_i(t)$ for 2nd-order model	17
β	no. of trains on track 2 of merging operation	67
<u>C(t)</u>	r.h.s. of Riccati equation	28
C	line capacity under min. headway	10
c_i	constant of proportionality between $\dot{f}_i(t)$ and $\dot{u}_i(t)$	15
c'_i	ratio c_i/m	15
d_0	constant drag force on each train	15
d_i	drag coeff. for ith train of 3rd-order model	15
d'_i	ratio d_i/m	15
<u>e(t)</u>	error state vector for train system	24
$\underline{e}_\gamma(t)$	error state vector for γ th merging sequence	68

<u>Symbol</u>	<u>Meaning</u>	<u>Symbol First Appears on Page</u>
$e_{i1}(t)$	error in position of i th train with respect to scheduled position	14
$e_{i2}(t)$	error in velocity of i th train with respect to scheduled velocity	14
$e_{i3}(t)$	time derivative of $e_{i2}(t)$	14
ϵ	difference between results of two successive iterations in solving the Riccati equation	23
$f_i(t)$	applied force of i th train	13
$\underline{G}$	coeff. matrix determining optimal trajectory	22
$g_i(x_i)$	drag force as a function of velocity for the i th train	13
γ	denotes γ th merging sequence	67
h	headway between successive trains	6
h_0	minimum headway for no collisions	10
J	quadratic performance criteria (cost functional)	21
J_A	general cost functional for 3rd-order model	18
J_B	general cost functional for 2nd-order model	20
J^*	optimal cost (minimum cost)	22
J_γ	cost for the γ th merging sequence	68
J_{γ^*}	cost for the optimal merging sequence γ^*	68
$\underline{K}(t)$	solution to the Riccati equation	22
$\hat{\underline{K}}$	steady state solution to the Riccati eqn.	22
$\hat{k}_{ij}$	element of $\hat{\underline{K}}$ corresponding to the i th row and j th column	29

<u>Symbol</u>	<u>Meaning</u>	<u>Symbol First Appears on Page</u>
k	ratio of min. headway to min. stopping distance	12
L	vehicle length	6
M	no. of possible merging sequences	67
m	mass of each train under equal mass assumption	14
m_i	mass of i th train	13
N	no. of trains in the system	6
n_c	no. of collisions in case of catastrophic failure	13
$\underline{Q}$	constant coeff. matrix within cost functional J	21
q_i	constant weighting factor for i th train in general cost functional	18
q'_i	constant weighting factor for i th train in general cost functional	18
$\underline{R}$	constant coeff. matrix within cost functional J	21
r_i	constant weighting factor for control energy expenditure of i th train in generalized cost functional	18
$\underline{S}$	constant matrix from which state con- trollability is determined	25
s_i	constant weighting factor for i th train in general cost functional	18
s'_i	constant weighting factor for i th train in general cost functional	18
σ	dummy variable (velocity) in expression for min. stopping distance as fct. of velocity	10
T	terminal time in the control time interval	21

<u>Symbol</u>	<u>Meaning</u>	<u>Symbol First Appears on Page</u>
t	time	9
Δt	total time delay in emergency stopping procedure	9
τ	dummy variable for time	37
$\underline{u}(t)$	m-component control vector	21
$u_i(t)$	control variable for ith train of 2nd- order system	15
$\underline{u}^*(t)$	optimal control vector	21
v_c	cruising velocity for the train system	10
v_{impact}	impact vel. between 1st and 2nd trains for collision	54
v_s	scheduled velocity of ith train	14
$\underline{w}(t)$	control vector for 3rd-order system	24
$w_i(t)$	control variable for ith train of 3rd- order system	16
$\underline{w}^*(t)$	optimal control vector for 3rd-order system	35
$\underline{x}(t)$	n-component state vector	21
$x_i(t)$	position of ith train	7
$x_{i1}(t)$	position state variable ($x_{i1}=x_i$)	14
$x_{i2}(t)$	velocity state variable ($x_{i2}=\dot{x}_i$)	14
$x_{i3}(t)$	acceleration state variable ($x_{i3}=\ddot{x}_i$)	14
$x_{i\gamma}$	position of ith train for γ th merging sequence	68
x_0	minimum stopping distance	10
$z_i(t)$	scheduled position of ith train	7

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1.1 The Balanced Transit System

For the past several years the problem of urban transportation has generally been attacked by the construction of vast freeway systems to facilitate the movement of private automobile traffic. The automobile continues to be a desired form of transportation because of its comfort, convenience and privacy. Most cities also have a well developed bus system which in general utilizes the same roads and freeways as private and commercial automobile traffic.

As shown by Michaels^[1], modern high-speed rail rapid transit is the most efficient and economical method known for moving large numbers of people quickly and comfortably. Further, a "balanced" transit system with rail transportation as its "core" and including bus and automobile traffic can be the most advantageous scheme for urban centres. This scheme would result in less congestion and less air pollution and hence benefit user and non-user alike.

An interesting comparison can be made between the efficiency of various forms of transportation in the following manner. We will assume that on a lane of freeway traffic there is an average of 1.5 persons per car and 2.5 sec. spacing between cars (180 feet at 50 mph). Then the lane is capable of moving 2160 passengers per hour. Now let us assume a train of 5 cars, each capable of carrying 100 passengers. A train separation of 1.5 min. would result in a flow rate of 20,000 people per hour or the equivalent of about 9 lanes of freeway. Further, in the case of buses, for 36 second spacing and 50 passengers per bus the flow rate would be 5,000 people per hour. Although a

number of assumptions are made in arriving at the above flow rates, they do indicate that a higher efficiency is attainable by a train system.

In high density areas train operation could be made even more efficient by operating with smaller headways (the distance from the rear of one train to the front of the following train is termed the headway). In the above discussion spacings were given in units of time which allows calculation of headway in units of distance for any velocity. A further discussion of headways is given in Chapter 2.

From the above it is clear that high speed rail lines provide transportation of highest efficiency in areas where the density of passengers is high enough to warrant short headways. Most of the existing high speed rail systems were built before the use of the automobile had become widespread. Hence they were restricted to urban areas of high population density such as New York City. Today in areas such as Los Angeles, the population density is not as high as in New York but the use of the automobile and bus as feeder systems for a centrally located rail system can contribute to an equally efficient operation. Such a balanced system could find applications in most North American cities with populations exceeding 400,000. This would include such Canadian cities as Winnipeg, Vancouver and Edmonton.

The last consideration which we will mention here is that of cost of the system. The cost of adding rail lines to existing freeway systems need not be high. For example, in Chicago, a rail line was constructed in the median of the 4 lane Eisenhower expressway at an increase in total expenditure of only 10%. In many areas, sections of existing rail line could be converted for use in a balanced transit system thus reducing the cost considerably. Certainly in many areas,

congestion has become so serious a problem as to make cost of a superior system less of a consideration.

One primary requirement for the successful operation of the high speed systems discussed above is the development of safe, reliable and effective automatic control systems for controlling vehicle speed, acceleration and spacing while running at the most efficient headway. Passenger comfort becomes a prime consideration and results in several constraints being placed on any envisioned control system. These will also be considered in Chapter 2.

1.2 Two Operational Systems

In this section two of the several existing high speed rail systems will be discussed briefly so as to indicate the specifications of these systems in regard to top speeds, headways, etc.

The Bay Area Rapid Transit (BART) System of San Francisco is perhaps the best example of a modern high speed rail system as envisioned in the previous section. Scheduled to begin revenue operations in 1970, the system has the following design standards:

1. The trains run on separated rights of way including aerial structures, subways, and underwater tubes which free the entire system from grade crossings.
2. Propulsion is provided by a 1000-volt d-c vehicle propulsion and power supply system based on a 1000-volt d-c contact rail and solid-state, chopper-controlled traction motors.
3. Straight line performance specifications are acceleration from zero to 50 mph in 20 seconds in contrast to the usual train standard of 20 seconds from zero to 25 mph. The top cruising speed is 80 mph.

4. A headway of about 90 seconds is maintained by the automatic train control system which is also in charge of all train movements such as acceleration, speed control and deceleration. Provision is also made for manual operation.

The primary principle of regulation used in the BART scheme is that all trains moving over a given section of track will be moving at the same speed. This will be a prime requirement of the control system described later in this thesis.

Another train system which has been in operation for a considerably longer time than the BART System is the New Tokaido Line of the Japanese National Railways. Commercial passenger service was opened between Tokyo and Osaka on October 1, 1964 and has since operated successfully with a traffic load of up to 150,000 passengers per day.

Since it is an inter-city system, the specifications of the Tokaido System differ considerably from those of the intra-city BART System described earlier.

1. The maximum operating speed is 130 mph with a minimum headway of 5 min.
2. Deceleration requires 5,600 feet to slow from 130 mph to 100 mph (first stage) and 5,250 feet to slow from 100 mph to zero (second stage). Including pre-braking coasting, it takes 11,600 feet to stop from a speed of 130 mph. The resultant maximum deceleration rate is 1.6 mph/sec.
3. The control system operates as follows: Each section of the line has a specified speed depending on the nearness of stations, curves and the presence of other trains on the line. On entering a section at other than the required

speed, acceleration or braking is applied automatically.

The normal length of a controlled block of track is 10,000 feet.

4. In the future, operational speeds of up to 160 mph are envisioned which according to Fujii^[2] is close to the maximum practical speed limit for a conventional railway of standard gauge.

Due to this practical speed limit much attention has been paid in the literature to new and often radical methods of propulsion and guidance. As we are primarily interested in the control of conventional systems, these will not be described here.

1.3 Scope of This Thesis

In the following chapters a high speed train regulator will be developed using existing theory of optimal control (the state regulator problem). A mathematical model of the system is developed from Newton's Second Law. The work reported is basically an extension of the work of Levine and Athans^{[3][4]} and differs in the following respects. Jerk (time rate of change of acceleration) is taken into account by differentiating the equation of motion for each train. Several performance criteria (in the form of cost functionals) are used to evaluate the optimal control system's performance. A comparative study of the system of Levine and Athans and that developed in this thesis is made by simulating both control systems on an analogue computer.

CHAPTER 2

PROBLEM FORMULATION

2.1 Description of the System and Objectives

The system of trains to be considered is represented in figure 2-1. There are N trains in the system of which only a typical section of three is shown in figure 2-1. The mass of the i th vehicle is m_i and its scheduled position at time t is given by $z_i(t)$. The headway between vehicles is h and the length of each vehicle is L .

In general, there are two ways of regulating the motion of a string of vehicles. Either each vehicle can be made to regulate its velocity and position with respect to an internal on-board reference, or a communications system can provide each vehicle with a knowledge of the velocity and position of every other vehicle in the string thereby enabling the regulation of differences in velocity and position of adjacent vehicles. It is fairly obvious that the first type of regulation would not prevent collisions in the case of uncontrolled motion of one or more vehicles in the string (as in the case of a de-railment). The second type of regulation will control the headway and for this reason we will deal primarily with regulation of this type.

2.2 Assumptions and Constraints

The following assumptions are made in formulating the mathematical model for the system:

1. It will be assumed that the system of vehicles is running on a flat section of straight guideway. This assumption is important in the derivation of the equation of motion for each vehicle.
2. The operation is assumed to take place far enough between

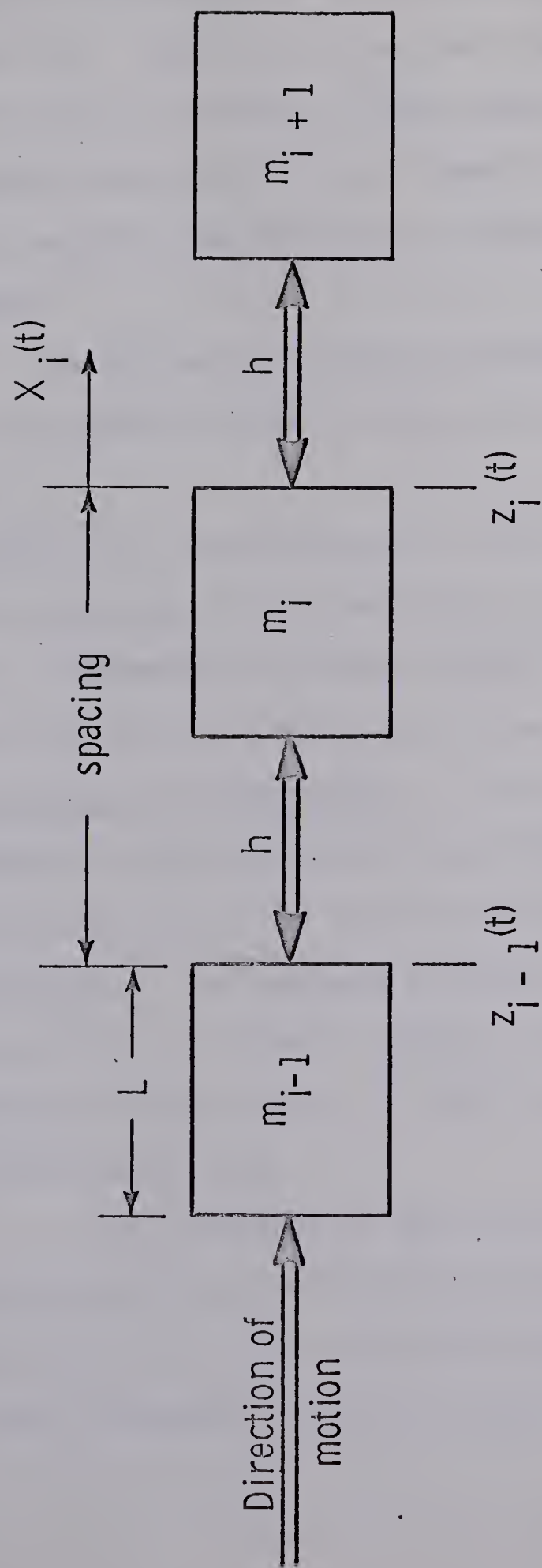


Figure 2-1

THE PHYSICAL SYSTEM

stations that controlled braking or acceleration is not required.

3. The vehicles making up the train are assumed to be of equal mass. This will in practise be realized only when each vehicle is carrying an equal number of passengers. Normally, there will not be too great a deviation from this condition when the system is operating in an efficient manner.

We are now in a position to consider the constraints imposed on the system which must be taken into account when formulating the problem.

1. Velocity - The operating speed of the system of trains is self-constrained by the limitations of the physical equipment such as propulsion and guidance devices and thus will not be considered in the formulation of the problem.
2. Acceleration and Deceleration - In the interest of passenger comfort we must constrain the acceleration forces generated by the controller to be less than some maximum figure. Friedlander^[5] has suggested a maximum limit equal to 1.4 times the pull of gravity. However, it is felt that under emergency conditions (i.e. to avert a collision) that higher forces could be used.

It is interesting to note that the maximum acceleration-deceleration rates of the BART and Tokaido Systems are both less than 0.1 g. which is well below that experienced in the private automobile (0.8 g. for a good disc brake system).

3. Jerk - Time rate of change of acceleration force or jerk force is more objectionable to passengers than the acceleration force itself.

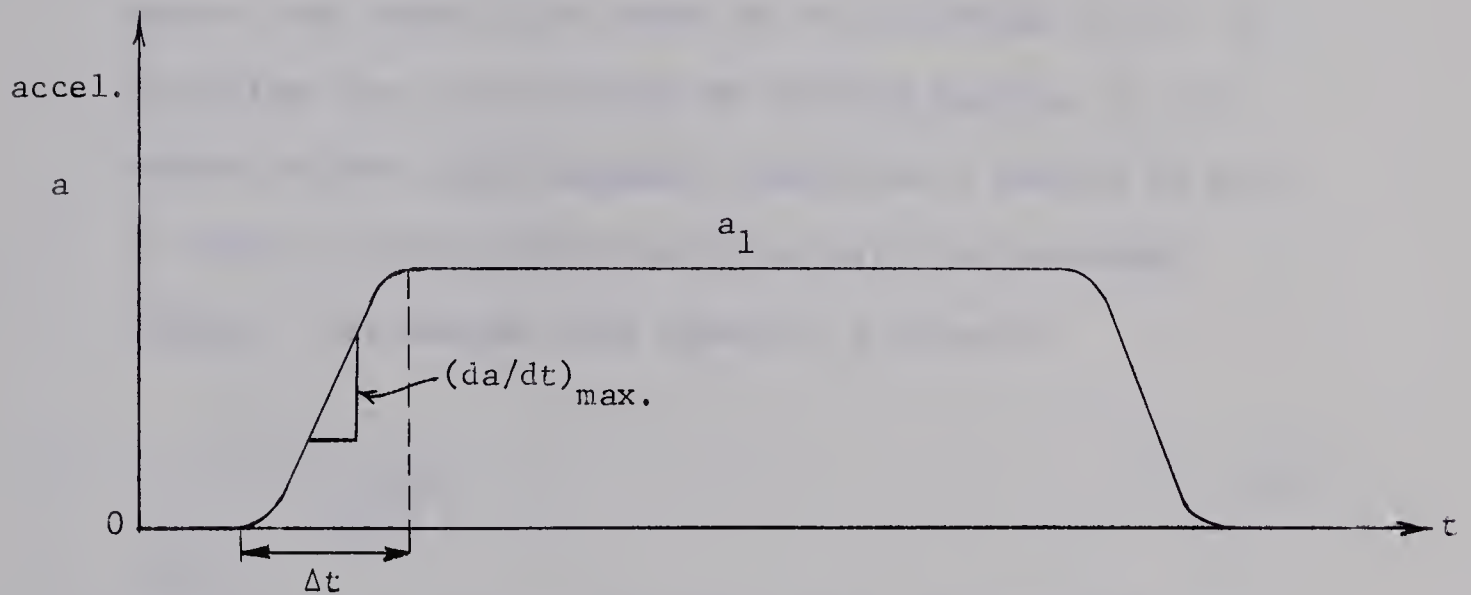


FIGURE 2-2 A POSSIBLE ACCELERATION TRAJECTORY

Figure 2-2 shows a possible plot of acceleration vs time for correction in velocity of a train in the system. The time delay Δt in application of acceleration a_1 is due to the sum of all time delays in the system including inertia of the vehicle. Depending on Δt , the maximum jerk given by $(da/dt)_{max}$ can be excessive. In the private automobile, the driver is in control of the applied jerk at all times and generally applies acceleration forces (except in emergency cases) in such a manner as to make the maximum jerk acceptable to him yet accomplishing his required velocity change in a reasonable length of time. This is exactly what we would like to accomplish with our train regulator.

4. Headway - For a number of reasons this parameter must impose a rigid constraint on our system. As indicated in figure 2-1, headway is defined as the distance from the rear of one train to the front of the following train. If collisions are to be avoided the minimum headway, h_0 , is chosen so that under emergency conditions a vehicle is able to come to a stop without colliding with the preceding vehicle. The maximum line capacity is given by

$$C = \frac{v_c}{h_0 + L} \quad (2.1)$$

where v_c is the cruising velocity and L the vehicle length.

Hojdu et al^[6] give the minimum stopping distance especially in the case of regenerative braking as

$$x_0 = \int_0^{v_c} \frac{\sigma d\sigma}{a(\sigma)} + \Delta t v_c \quad (2.2)$$

where v_c is the velocity before deceleration and $a(\sigma)$ expresses deceleration as a function of the train velocity (σ is a dummy variable). The total communication and reaction time delay is given by Δt .

Figure 2-3a shows an idealized trapezoidal deceleration trajectory. The corresponding velocity trajectory is given in figure 2-3b. By expressing the deceleration in terms of the velocity at any time it has been shown^[6] that by applying 2.2, the minimum stopping distance is

$$x_0 = \frac{v_c^2}{2a_M} + \frac{1}{2} \frac{a_M}{\frac{da}{dt}} v_c + \Delta t v_c \quad (2.3)$$

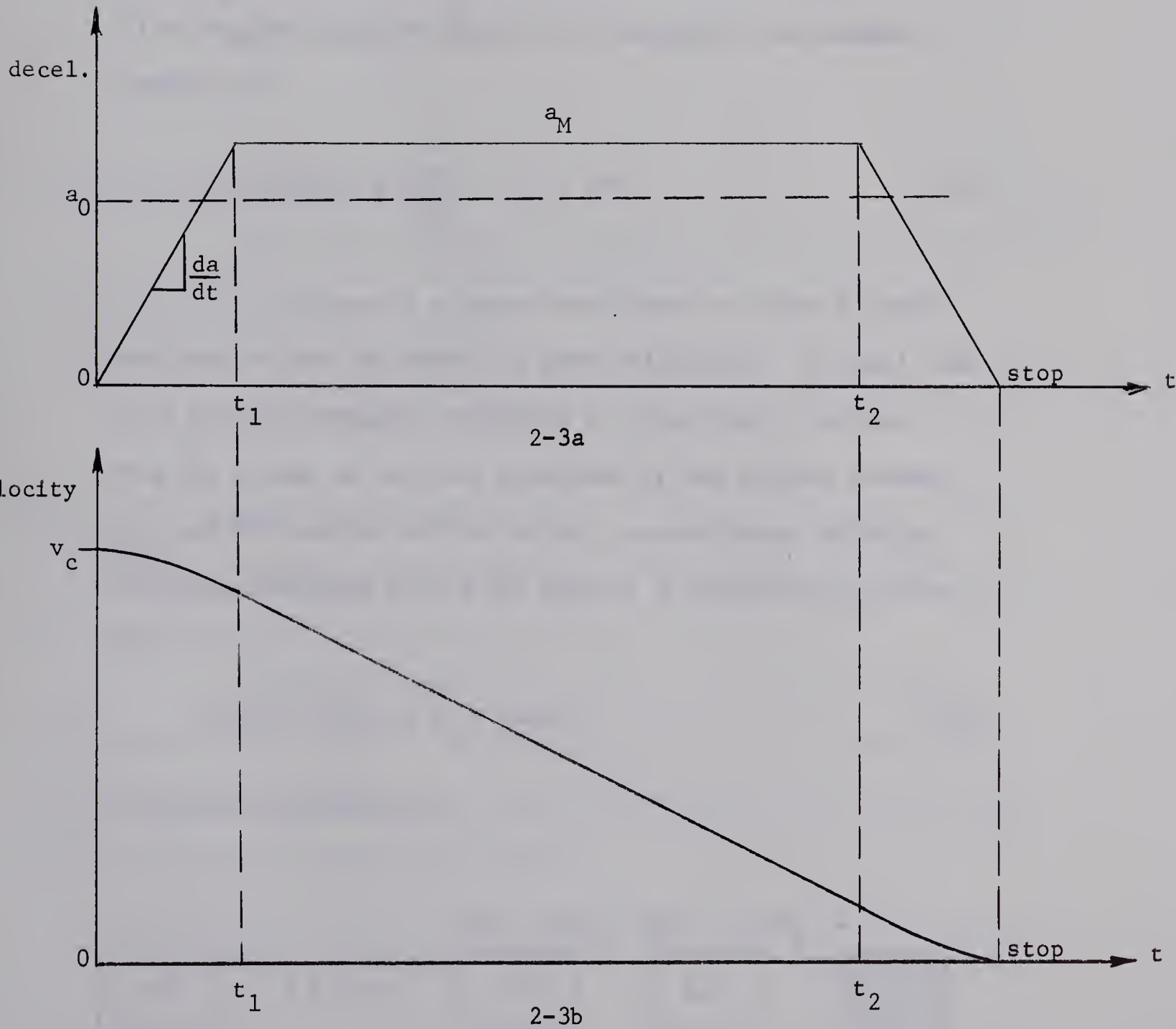


FIGURE 2-3 DECELERATION AND VELOCITY BRAKING TRAJECTORIES

If a fictitious constant emergency deceleration rate $a_0 < a_M$ is defined then we can write

$$x_0 = \frac{v_c^2}{2a_0} + \Delta t v_c \quad (2.4)$$

where a_0 reflects the effects of maximum allowable jerk.

If we neglect the time delay Δt we can write the minimum headway as

$$h_0 = kx_0 = k \frac{v_c^2}{2a_0}, \quad k > 0 \quad (2.5)$$

In the case of a train de-railment or track blockage and with $k < 1$ we can expect to have collisions. We shall term this type of emergency condition a "catastrophic failure".

For the string of vehicles separated by the minimum headway h_0 , the Nth vehicle will be able to stop without colliding with the preceding car in the case of a catastrophic failure if

$$Nh_0 + (N-1)L \geq x_0 + (N-1)L \quad (2.6)$$

as shown in figure 2-4.

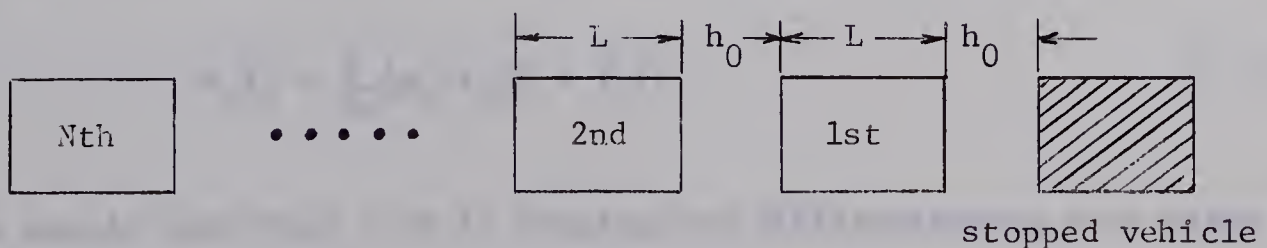


FIGURE 2-4 A CATASTROPHIC FAILURE

Equation 2.6 can be reduced to

$$Nh_0 \geq x_0 = \frac{h_0}{k} \quad (2.7)$$

Then the number of trains that will collide in the case of a catastrophic failure is

$$n_c = N-1 = \frac{1}{k} - 1, \quad 0 < k \leq 1 \quad (2.8)$$

2.3 Mathematical Model of the System

Given the assumptions of section 2.2, Levine and Athans^{[3][4]} describe the motion of the i th train in a string of N trains by the second-order differential equation

$$m_i \ddot{x}_i + g_i(\dot{x}_i) = f_i(t) \quad (2.9)$$

obtained by applying Newton's Second Law. The mass of the i th train is given by m_i and its displacement by x_i . The drag force on the i th train is described by $g_i(\dot{x}_i)$ and the applied force by $f_i(t)$.

As already stated, jerk ($\ddot{\dot{x}}_i$) is an important factor affecting passenger comfort. In order to make jerk a variable in the system equation it is proposed to describe the motion of the i th train by the third-order differential equation

$$m_i \ddot{\dot{x}}_i + \frac{d}{dt}[g_i(\dot{x}_i)] = \dot{f}_i(t) \quad (2.10)$$

It is easily seen that 2.10 is obtained by differentiating both sides of 2.9 with respect to time.

Since we have assumed equal mass for all trains we shall call $m_i = m$ for $i=1, \dots, N$. Then 2.10 can be written

$$\ddot{x}_i + \frac{\dot{g}_i(\dot{x}_i)}{m} = \frac{\dot{f}_i(t)}{m} \quad (2.11)$$

By defining $x_{i1} = x_i$, $\dot{x}_{i1} = \dot{x}_i$ and $\ddot{x}_{i1} = \ddot{x}_i$ as the state variables we get the following state equations:

$$\begin{aligned} \dot{x}_{i1}(t) &= x_{i2}(t) \\ \dot{x}_{i2}(t) &= x_{i3}(t) \\ \dot{x}_{i3}(t) &= -\frac{\dot{g}_i(x_{i2})}{m} + \frac{\dot{f}_i(t)}{m} \end{aligned} \quad (2.12)$$

Now we can define error state variables as follows:

$$\begin{aligned} e_{i1}(t) &= x_{i1}(t) - z_i(t) \\ e_{i2}(t) &= x_{i2}(t) - v_s \\ e_{i3}(t) &= x_{i3}(t) \end{aligned} \quad (2.13)$$

where $z_i(t)$ is the scheduled position of the i th train at time t , v_s is the scheduled velocity of the i th train (the cruising velocity) and is constant.

Differentiating 2.13 and substituting 2.12 yields

$$\dot{e}_{i1}(t) = \dot{x}_{i1}(t) - \dot{z}_i(t) = \dot{x}_{i1}(t) - v_s = e_{i2}(t) \quad (2.14)$$

$$\dot{e}_{i2}(t) = \dot{x}_{i2}(t) - \dot{v}_s = \dot{x}_{i2} = e_{i3}(t) \quad (2.15)$$

$$\dot{e}_{i3}(t) = \dot{x}_{i3}(t) = -\frac{\dot{g}_i(x_{i2})}{m} + \frac{\dot{f}_i(t)}{m} \quad (2.16)$$

Taken together 2.14, 2.15 and 2.16 define a new set of state equations describing the system.

As a linear approximation of the drag force let

$$g_i(x_{i2}) = d_0 + d_i e_{i2}(t) \quad (2.17)$$

where d_0 is the constant drag when the velocity of the i th train equals the scheduled velocity. Then

$$\dot{g}_i(x_{i2}) = d_i \dot{e}_{i2}(t) = d_i e_{i3}(t) \quad (2.18)$$

Now 2.16 becomes

$$\dot{e}_{i3}(t) = \frac{-d_i e_{i3}(t) + \dot{f}_i(t)}{m} \quad (2.19)$$

and if we let $d'_i = d_i/m$, $c'_i = \dot{f}_i(t)/\dot{u}_i(t)$ and $c'_i = c_i/m$ where $u_i(t)$ is the control variable for the i th train we can write

$$\dot{e}_{i3}(t) = -d'_i e_{i3}(t) + c'_i \dot{u}_i(t) \quad (2.20)$$

Calling $\dot{u}_i(t) = w_i(t)$ we can write the final state equations for the i th train as

$$\begin{aligned}\dot{e}_{i1}(t) &= e_{i2}(t) \\ \dot{e}_{i2}(t) &= e_{i3}(t) \\ \dot{e}_{i3}(t) &= -d'_i e_{i3}(t) + c'_i w_i(t)\end{aligned}\tag{2.21}$$

Written in matrix form 2.21 becomes

$$\begin{bmatrix} \dot{e}_{i1} \\ \dot{e}_{i2} \\ \dot{e}_{i3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -d'_i \end{bmatrix} \begin{bmatrix} e_{i1} \\ e_{i2} \\ e_{i3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c'_i \end{bmatrix} w_i(t)\tag{2.22}$$

or

$$\dot{\mathbf{e}}_i = \mathbf{A}_i \mathbf{e}_i + \mathbf{B}_i w_i(t)\tag{2.23}$$

Note that $w_i(t)$ is directly related to the jerk forces applied to the i th train. Specifically,

$$w_i(t) = \frac{d}{dt} \frac{f_i(t)}{mc'_i} = \frac{\dot{f}_i(t)}{c_i}\tag{2.24}$$

For the sake of comparison, if the system of equations developed by Levine and Athans is written in a form similar to that of

2.22 we obtain

$$\begin{bmatrix} \dot{e}_{i1} \\ \dot{e}_{i2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -a_i \end{bmatrix} \begin{bmatrix} e_{i1} \\ e_{i2} \end{bmatrix} + \begin{bmatrix} 0 \\ b_i \end{bmatrix} u_i(t) \quad (2.25)$$

where

$$a_i = \frac{g_i(x_{i2}) - d_0}{e_{i2}(t)}, \quad b_i = \frac{f_i(t) - d_0}{u_i(t)}$$

2.4 Cost Functionals

Thus far, we have not indicated in what manner we will derive a regulator for the system of trains. From the discussion of sections 2.1 and 2.2 it is evident that we wish to minimize certain deviations from desired responses while constraining acceleration, jerk and headway. Since it is our intention to use available techniques in optimal control theory, it is necessary to express our objectives in the form of a performance index or cost functional. In this section we shall develop a general cost functional which will express our objectives in a mathematical form.

It was mentioned earlier that there are two ways of regulating a variable such as say the error in position with respect to scheduled position, e_{i1} . If we minimize e_{i1} for all trains in the system, each train would need only an on-board reference giving the current scheduled position at all times. If, on the other hand, we minimize $(e_{i1} - e_{i+1,1})$ for all trains then each train requires a

knowledge of the position error of all other trains at all times. Since the second type of minimization would likely result in a more stable system we shall concentrate on difference terms when setting down cost functionals.

The most general cost functional considered is given by

$$J_A = \int_0^{\infty} \left[\sum_{i=1}^N r_i w_i^2(t) + \sum_{i=1}^{N-1} q_i (e_{i1} - e_{i+1,1})^2 + \sum_{i=1}^{N-1} s_i (e_{i2} - e_{i+1,2})^2 + \sum_{i=1}^N q'_i e_{i1}^2 + \sum_{i=1}^N s'_i e_{i2}^2 \right] dt \quad (2.26)$$

where N is the total number of trains and $r_i, q_i, q'_i, s_i, s'_i$ are non-negative weighting factors.

The following comments on 2.26 will explain the significance of the various terms in the cost functional.

1. All terms in the functional appear as quadratics, so that large deviations are penalized more heavily than are small deviations. This is a reasonable requirement since we wish the system to operate as close to the schedule as possible.
2. The first term penalizes the system for excessive jerk forces and hence reflects the passenger comfort criteria established in section 2.2.
3. The second term penalizes the system for differences in position error between successive trains. This has the effect of penalizing the system for deviations from the scheduled headway.

4. The third term penalizes the system for differences in velocity error between successive trains. At this time it would seem that the addition of the third term will have little effect on the behaviour of the system. That is, if two successive trains are running with no difference in position error between them then their separation is constant. Hence they must be running at the same speed so that the difference in velocity error between them would be zero. If only the third term was included, however, it would be expected that the intervehicle spacing would not be equal for all pairs of trains but the spacing should remain constant.

5. The second and third terms do not guarantee that either the position or velocity errors of the individual trains will be zero. The terms e_{il}^2 and $e_{i+1,l}^2$ can be large as long as $(e_{il} - e_{i+1,l})^2$ remains close to zero.

To penalize the system for deviations from the schedule, the fourth and fifth terms in 2.24 are inserted.

6. The upper limit on the integral is specified as ∞ in order to ensure that the cost stays near zero after an initial transient interval and to avoid the specification of a large terminal time. Even if a large terminal time were chosen we could not guarantee that the cost would remain near zero for time greater than the specified terminal time.

7. The cost functional is symmetrical in that positive and negative jerk and deviations from position and velocity in either direction are penalized equally.

8. The constraints mentioned in section 2.2 are implied only in as much as an increase in weighting of a particular term in 2.26 has the effect of constraining it.

9. For the second-order system described by equation 2.25, the general cost functional used is very similar to that of 2.26 except the control u_i is substituted for w_i . That is,

$$J_B = \int_0^{\infty} \left[\sum_{i=1}^N u_i^2(t) + \sum_{i=1}^{N-1} q_i (e_{i1} - e_{i+1,1})^2 + \sum_{i=1}^{N-1} s_i (e_{i2} - e_{i+1,2})^2 + \sum_{i=1}^N q'_i e_{i1}^2 + \sum_{i=1}^N s'_i e_{i2}^2 \right] dt \quad (2.27)$$

10. The above cost functionals do not in any way ensure good dynamic behaviour of the regulator under all possible conditions and each system must be simulated on a computer in order to check the effect of the various disturbances on the response of the system.

In the following chapter several different cost functionals of the above forms are used to specify optimal regulators for both the second and third-order representations of the train system. In doing so we provide a wider basis of comparison between the two representations.

CHAPTER 3 DEVELOPMENT OF THE OPTIMAL CONTROLLER

3.1 Method of Solution

Levine and Athans have shown^{[3][4]} that the problem outlined in the preceding chapter can be solved by the method that has been developed for the optimal control of a linear system with a quadratic performance criterion. This problem is often referred to as the linear regulator problem.

Although the details of this method are available in the literature and textbooks (for example see [7]), a summary of the relevant results is presented here for convenience.

Consider a linear time-invariant system described by the vector matrix differential equation

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) \quad (3.1)$$

where $\underline{x}(t)$ is the n element column state vector, $\underline{u}(t)$ the m element column control vector and $\underline{A}$ and $\underline{B}$ are $n \times n$ and $n \times m$ constant matrices respectively. The system is assumed to be controllable (see Appendix 1 for definition of controllability).

The objective is to obtain a control $\underline{u}^*(t)$ which will take the state $\underline{x}(t)$ from an initial state $\underline{x}(t_0)$ to a final state $\underline{x}(T)$ while minimizing the functional

$$J = \frac{1}{2} \int_0^\infty [\langle \underline{x}(t), \underline{Q}\underline{x}(t) \rangle + \langle \underline{u}(t), \underline{R}\underline{u}(t) \rangle] dt \quad (3.2)$$

where $\underline{Q}$ is a positive semi-definite constant matrix and $\underline{R}$ is a positive definite constant matrix. The control $\underline{u}(t)$ is unconstrained.

It has been shown that the optimal control exists, is unique and is given by

$$\underline{u}^*(t) = -\underline{R}^{-1}\underline{B}'\hat{\underline{K}}\underline{x}(t) \quad (3.3)$$

where the superscript prime denotes transpose and $\hat{\underline{K}}$ is the constant nxn positive definite matrix which is the solution of the nonlinear matrix algebraic equation

$$-\hat{\underline{K}}\underline{A} - \underline{A}'\hat{\underline{K}} + \hat{\underline{K}}\underline{B}\underline{R}^{-1}\underline{B}'\hat{\underline{K}} - \underline{Q} = \underline{0} \quad (3.4)$$

The optimal trajectory is the solution of the linear time-invariant homogeneous system

$$\dot{\underline{x}}(t) = \underline{G}\underline{x}(t) \quad , \quad \underline{x}(0) \text{ given} \quad (3.5)$$

where $\underline{G}$ is defined by

$$\underline{G} = \underline{A} - \underline{B}\underline{R}^{-1}\underline{B}'\hat{\underline{K}}$$

The minimum cost J^* is given by

$$J^*[\underline{x}(t)] = \frac{1}{2} \langle \underline{x}(t), \hat{\underline{K}}\underline{x}(t) \rangle \quad (3.6)$$

Concerning the matrix $\hat{\underline{K}}$, it has been shown^[8] that the assumptions of controllability and the terminal cost being zero imply that

$$\hat{\underline{K}} = \lim_{T \rightarrow \infty} \underline{K}(t) \quad (3.7)$$

where the limit exists and is unique and $\underline{K}(t)$ is the solution to the Riccati equation

$$\dot{\underline{K}}(t) = -\underline{K}(t)\underline{A} - \underline{A}'\underline{K}(t) + \underline{K}(t)\underline{B}\underline{R}^{-1}\underline{B}'\underline{K}(t) - \underline{Q} \quad (3.8)$$

with boundary condition $\underline{K}(T) = \underline{Q}$ where T is the final time.

Athans and Falb^[7] give the following interpretation to 3.8. If the time T is thought of as the "starting time" and $\underline{K}(T)$ as the "initial condition", then the matrix $\hat{\underline{K}}$ can be thought of as the "steady-state" solution of the Riccati equation as the time decreases. As $T \rightarrow \infty$, the "transient" due to the "initial condition" dies down and the solution approaches the value $\hat{\underline{K}}$ for $t \ll T$.

Equation 3.8 can hence be solved by a numerical method by iterating backwards in time starting at some large T . The problem of deciding when the steady state value has been reached (i.e. when to stop iterating) is usually solved by deciding upon some ϵ such that when the results of two successive iterations differ by ϵ for the first time the procedure is stopped. Deciding on an ϵ can be done by observing the results of the iterative scheme and in practice this should present no difficulties.

3.2 Optimal Control For the Train System

The method described in the preceding section will now be applied to the system of trains. We shall carry out the investigation for a string of three trains so as to keep the number of equations to a reasonable level. This choice will also enable us to compare or check our results with those obtained by Levine and Athans

since they also consider a string of three vehicles.

For three vehicles 2.22 becomes

$$\begin{bmatrix} \dot{e}_{11} \\ \dot{e}_{12} \\ \dot{e}_{13} \\ \dot{e}_{21} \\ \dot{e}_{22} \\ \dot{e}_{23} \\ \dot{e}_{31} \\ \dot{e}_{32} \\ \dot{e}_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d'_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -d'_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d'_3 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \\ e_{21} \\ e_{22} \\ e_{23} \\ e_{31} \\ e_{32} \\ e_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c'_1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c'_2 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c'_3 \end{bmatrix} \begin{bmatrix} w_1 \\ \hline w_2 \\ \hline w_3 \end{bmatrix} \quad (3.9)$$

This can be written

$$\dot{\underline{e}}(t) = \underline{A}\underline{e}(t) + \underline{B}\underline{w}(t) \quad (3.10)$$

where $\underline{A}$ and $\underline{B}$ are the 9x9 and 9x3 matrices of 3.9 respectively. The partitioning of 3.9 makes it easier to see the coefficients associated with each train in the system. Note that 3.10 is of the same form as 3.1. The cost functional in 2.26 can be written as

$$J = \frac{1}{2} \int_0^\infty [\langle \underline{e}(t), \underline{Q}\underline{e}(t) \rangle + \langle \underline{w}(t), \underline{R}\underline{w}(t) \rangle] dt \quad (3.11)$$

where

$$\underline{Q} = \begin{bmatrix} q_1' + q_1 & 0 & 0 & -q_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_1' + s_1 & 0 & 0 & -s_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q_1 & 0 & 0 & q_1 + q_2 + q_2' & 0 & 0 & -q_2 & 0 & 0 \\ 0 & -s_1 & 0 & 0 & s_1 + s_2 + s_2' & 0 & 0 & -s_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_2 & 0 & 0 & q_2 + q_3' & 0 & 0 \\ 0 & 0 & 0 & 0 & -s_2 & 0 & 0 & s_2 + s_3' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.12)$$

and

$$\underline{R} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \quad (3.13)$$

As can be seen, both $\underline{Q}$ and $\underline{R}$ are symmetric. Observing that the second, third, fourth, and fifth summations of 2.26 are ≥ 0 we can say that $\langle \underline{e}(t), \underline{Q}\underline{e}(t) \rangle \geq 0$ hence $\underline{Q}$ is positive semi-definite. Also it can be seen that $\underline{R}$ is positive definite.

Lastly, we must verify that the system of 3.9 is completely controllable. We can do this by evaluating $\underline{S}$ of A1.3 (see Appendix 1) for the entire system or for each train in the string (i.e. show that 2.22 is controllable). Evaluating $\underline{S}$ for 2.22 gives

$$\underline{S} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad (3.14)$$

It can be verified by observation that the 3 column vectors of 3.14 are linearly independent. This also can be verified by observing that $\det \underline{S} = -1$ which implies $\underline{S}$ is of rank 3.

Hence we can proceed to solve the Riccati equation for the system of 3.9 obtained by using the cost functional of 3.11.

Due to time limitations it was decided to set $d'_i = \frac{d_i}{m} = c'_i = \frac{\dot{f}_i(t)}{\dot{u}_i(t)_m} = 1$ for all i and to concentrate on varying the weighting factors within the cost functional in order to evaluate performance of the system. This is a somewhat arbitrary decision but if we work on a unit mass basis and take $f_i(t) = u_i(t)$ and assume that the drag force on each train is $d_0 + e_{i2}(t)$ then we have $d'_i = c'_i = 1$ which will at least let us evaluate regulator performance.

With d'_i and c'_i set equal to 1 we can substitute into the Riccati equation 3.8 for some values of q_i , r_i , s_i , q'_i and s'_i . In choosing weighting factors we are really concerned with relative weighting, hence the $1/2$ in front of the cost functional can be absorbed into the weighting factors and r_i set equal to 1 for all i . Also, since we wish to penalize the system equally for the misbehaviour of any train, we can set $q_i = q$, $q'_i = q'$, $s_i = s$, $s'_i = s$ for all i .

The second-order system of 2.25 can be treated in a similar manner and a_i and b_i set equal to 1. We are now in a position to solve the Riccati equation for both the second and third-order systems

employing the following cost functionals:

$$J_1 = \int_0^\infty \left[\sum_{i=1}^3 w_i^2(t) + \sum_{i=1}^2 \{e_{i1}(t) - e_{i+1,1}(t)\}^2 \right] dt \quad (3.15)$$

$$J_2 = \int_0^\infty \left[\sum_{i=1}^3 w_i^2(t) + 5 \sum_{i=1}^2 \{e_{i1}(t) - e_{i+1,1}(t)\}^2 \right] dt \quad (3.16)$$

$$J_3 = \int_0^\infty \left[\sum_{i=1}^3 w_i^2(t) + 10 \sum_{i=1}^2 \{e_{i1}(t) - e_{i+1,1}(t)\}^2 \right] dt \quad (3.17)$$

$$J_4 = \int_0^\infty \left[\sum_{i=1}^3 w_i^2(t) + 20 \sum_{i=1}^2 \{e_{i1}(t) - e_{i+1,1}(t)\}^2 \right] dt \quad (3.18)$$

$$J_5 = \int_0^\infty \left[\sum_{i=1}^3 w_i^2(t) + 5 \sum_{i=1}^2 \{e_{i1}(t) - e_{i+1,1}(t)\}^2 \right. \\ \left. + 5 \sum_{i=1}^2 \{e_{i2}(t) - e_{i+1,2}(t)\}^2 \right] dt \quad (3.19)$$

$$J_6 = \int_0^\infty \left[\sum_{i=1}^3 w_i^2(t) + 10 \sum_{i=1}^2 \{e_{i1}(t) - e_{i+1,1}(t)\}^2 \right. \\ \left. + 10 \sum_{i=1}^2 \{e_{i2}(t) - e_{i+1,2}(t)\}^2 \right] dt \quad (3.20)$$

$$J_7 = \int_0^\infty \left[\sum_{i=1}^3 w_i^2(t) + 10 \sum_{i=1}^2 \{e_{i2}(t) - e_{i+1,2}(t)\}^2 \right] dt \quad (3.21)$$

$$J_8 = \int_0^\infty \left[\sum_{i=1}^3 w_i^2(t) + 10 \sum_{i=1}^3 e_{i2}^2(t) + 10 \sum_{i=1}^2 \{e_{i1}(t) - e_{i+1,1}(t)\}^2 \right] dt \quad (3.22)$$

$$J_9 = \int_0^\infty \left[\sum_{i=1}^3 w_i^2(t) + 10 \sum_{i=1}^3 e_{i1}^2(t) + 10 \sum_{i=1}^2 \{e_{i1}(t) - e_{i+1,1}(t)\}^2 \right] dt \quad (3.23)$$

In the case of the second-order system $u_i(t)$ is substituted for $w_i(t)$ in the above.

For each of the cost functionals, 3.8 can be evaluated and put in the form

$$\dot{\underline{K}}(t) = \underline{C}(t) \quad (3.24)$$

where $\underline{C}(t)$ is the right hand side of 3.8. Equation 3.24 represents nxn simultaneous nonlinear first-order differential equations but since $\hat{\underline{K}}$ is symmetric, we need only solve $\frac{n(n+1)}{2}$ simultaneous equations. Solution to 2.34 was obtained with the help of an IBM 360/67 digital computer using subroutine RKGS (fourth order Gill's method) in the "backwards mode". Results for the various functionals for second and third-order systems are shown in appendix 2.

For the sake of discussing the form of these results, the solution for the third-order system with cost J_3 is shown below.

$$\begin{bmatrix} 12.876 & -8.365 & 2.494 & -11.636 & -6.770 & -1.826 & -1.241 & -1.599 & -0.668 \\ -8.365 & 8.463 & 3.282 & -6.770 & -6.064 & -2.137 & -1.599 & -2.441 & -1.183 \\ 2.494 & 3.282 & 1.515 & -1.826 & -2.137 & -0.914 & -0.668 & -1.183 & -0.636 \\ -11.636 & -6.770 & -1.826 & 23.270 & 13.537 & 3.650 & -11.636 & -6.770 & -1.826 \\ -6.770 & -6.064 & -2.137 & 13.537 & 12.086 & 4.236 & -6.770 & -6.064 & -2.137 \\ -1.826 & -2.137 & -0.914 & 3.650 & 4.236 & 1.794 & -1.826 & -2.137 & -0.914 \\ -1.241 & -1.599 & -0.668 & -11.636 & -6.770 & -1.826 & 12.876 & 8.365 & 2.494 \\ -1.599 & -2.441 & -1.183 & -6.770 & -6.064 & -2.137 & 8.365 & 8.463 & 3.782 \\ -0.668 & -1.183 & -0.636 & -1.826 & -2.137 & -0.914 & 2.494 & 3.782 & 1.515 \end{bmatrix} \quad (3.25)$$

The above matrix is partitioned for the purpose of the following analysis and can be written as

$$\hat{\underline{K}} = \begin{bmatrix} \hat{\underline{K}}_{11} & \hat{\underline{K}}_{14} & \hat{\underline{K}}_{17} \\ \hat{\underline{K}}_{41} & \hat{\underline{K}}_{44} & \hat{\underline{K}}_{47} \\ \hat{\underline{K}}_{71} & \hat{\underline{K}}_{74} & \hat{\underline{K}}_{77} \end{bmatrix} \quad (3.26)$$

where

$$\hat{\underline{K}}_{ij} = \begin{bmatrix} \hat{k}_{ij} & \hat{k}_{i,j+1} & \hat{k}_{i,j+2} \\ \hat{k}_{i+1,j} & \hat{k}_{i+1,j+1} & \hat{k}_{i+1,j+2} \\ \hat{k}_{i+2,j} & \hat{k}_{i+2,j+1} & \hat{k}_{i+2,j+2} \end{bmatrix} \quad (3.27)$$

From observation of 3.25 there are three types of symmetry present:

1. symmetry about the principal diagonal
2. symmetry about the other diagonal
3. symmetry of each $\hat{\underline{K}}_{ij}$

As suggested by Levine and Athans^[4] the $\hat{\underline{K}}$ matrix can be thought of as a "coupling" matrix (see equation 3.3). Using this notion, the $\hat{\underline{K}}_{ij}$'s can be interpreted as follows:

- $\hat{\underline{K}}_{11}$ represents the self coupling of train 1
- $\hat{\underline{K}}_{44}$ represents the self coupling of train 2
- $\hat{\underline{K}}_{77}$ represents the self coupling of train 3
- $\hat{\underline{K}}_{14}$ and $\hat{\underline{K}}_{41}$ couple the 1st and 2nd trains
- $\hat{\underline{K}}_{17}$ and $\hat{\underline{K}}_{71}$ couple the 1st and 3rd trains
- $\hat{\underline{K}}_{47}$ and $\hat{\underline{K}}_{74}$ couple the 2nd and 3rd trains

With this interpretation in mind the three symmetries of 3.26 can be explained as follows:

1. $\underline{K}$ is symmetrical (about the principal diagonal) since the effect of say the 1st train on the 2nd train ($\hat{K}_{41}$) is the same as the effect of the 2nd train on the 1st ($\hat{K}_{14}$). This is reasonable and what we would expect.
2. K is symmetrical about the other diagonal since the system is symmetrical about the middle train. For example $\hat{K}_{14} = \hat{K}_{47}$ because the coupling between the 1st and 2nd trains is the same as that between the 2nd and 3rd trains. Again this is a reasonable outcome and reflects the symmetry of the cost functional.
3. The self-coupling matrices are symmetrical because the $\hat{K}$ matrix is symmetrical. Further $\hat{K}_{11} = \hat{K}_{79}$ again implies symmetry about the middle train. Internal symmetry of the off-diagonal submatrices can be explained by the following example.

$$\hat{K}_{17} = \begin{bmatrix} \hat{k}_{17} & \hat{k}_{18} & \hat{k}_{19} \\ \hat{k}_{27} & \hat{k}_{28} & \hat{k}_{29} \\ \hat{k}_{37} & \hat{k}_{38} & \hat{k}_{39} \end{bmatrix} \quad (3.28)$$

Here $\hat{k}_{18}$ represents the effect of the velocity of the 3rd train on the position of the 1st train and $\hat{k}_{27}$ represents the effect of the position of the 3rd train on the velocity of the 1st train. The fact that $\hat{k}_{18} = \hat{k}_{27}$ indicates an internal symmetry between the system state variables. Similar arguments explain why $\hat{k}_{19} = \hat{k}_{37}$ and $\hat{k}_{29} = \hat{k}_{38}$.

The above discussion simplifies the solution of the Riccati equation for other systems of this type since the number of equations which need to be solved is reduced.

3.3 Implementation of the Results

From equations 3.3 and 3.5 we can construct the closed loop optimal control system as shown below.

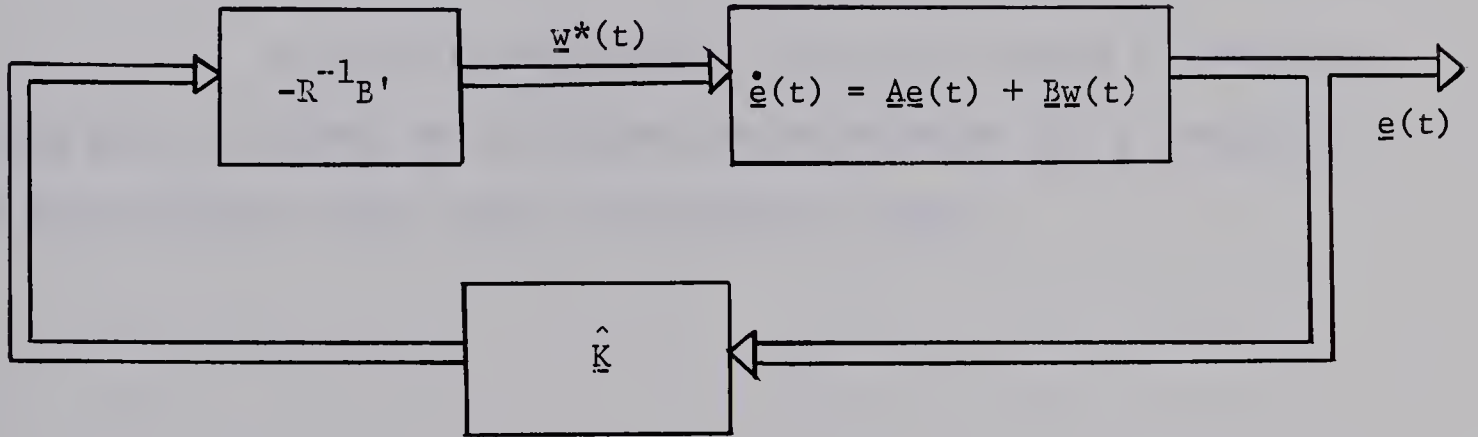


FIGURE 3-1 THE CLOSED LOOP OPTIMAL REGULATOR

From 3.3

$$\underline{w}^*(t) = -\underline{R}^{-1}\underline{B}'\hat{\underline{K}}\underline{e}(t) \quad (3.29)$$

Substituting for $\underline{R}$ and $\underline{B}$ we obtain

$$\underline{w}^*(t) = - \begin{bmatrix} \hat{k}_{31}e_{11}(t) + \hat{k}_{32}e_{12}(t) + \dots + \hat{k}_{39}e_{33}(t) \\ \hat{k}_{61}e_{11}(t) + \quad \cdot \quad \cdot \quad \cdot \quad + \hat{k}_{69}e_{33}(t) \\ \hat{k}_{91}e_{11}(t) + \quad \cdot \quad \cdot \quad \cdot \quad + \hat{k}_{99}e_{33}(t) \end{bmatrix} \quad (3.30)$$

Note that due to the nature of the $\underline{B}$ matrix only the 3rd, 6th and 9th rows of $\hat{\underline{K}}$ determine the optimal control $\underline{w}^*(t)$. In a similar manner we obtain the optimal control for the second-order system as

$$u^*(t) = \begin{bmatrix} \hat{k}_{21}e_{11}(t) + \hat{k}_{22}e_{12}(t) + \dots + \hat{k}_{26}e_{32}(t) \\ \hat{k}_{41}e_{11}(t) + \quad \cdot \quad \cdot \quad \cdot \quad + \hat{k}_{46}e_{32}(t) \\ \hat{k}_{61}e_{11}(t) + \quad \cdot \quad \cdot \quad \cdot \quad + \hat{k}_{66}e_{32}(t) \end{bmatrix} \quad (3.31)$$

In the following chapter the optimal system is simulated using both the second and third-order representations and a comparison of the behaviour under various disturbances is made.

CHAPTER 4

ANALOGUE COMPUTER SIMULATION OF THE
TRAIN SYSTEM

4.1 Introduction

Using the solutions to the Riccati equations corresponding to the various cost functionals obtained in the previous chapter and using the closed loop configuration shown in figure 3-1 both the second and third-order models of the train system were simulated on a PACE 231R analogue computer. The analogue computer diagrams for the second and third-order models are shown in figures 4-1 and 4-2 respectively. In this chapter, comparisons in the performance of the optimal control systems for the second-order and third-order models using the different cost functionals are presented.

In order to assess the usefulness of the two models they were both subjected to the same series of tests which are outlined below:

1. Train #1 is initially going too fast. This condition is simulated by setting positive initial conditions on the velocity and position errors for the first train. This test was performed on both the second and third-order systems for the cost functionals J_1 , J_3 , J_4 , J_6 , J_7 and J_9 defined in chapter 3. In the work of Levine and Athans^{[3][4]} only the simulation of the second-order train system is considered and that for only one cost functional (J_3).
2. A small step change in the velocity error of train #2. This is the most interesting test in so far as comparing the second and third-order systems is concerned in that the relationship between the respective jerk and acceleration

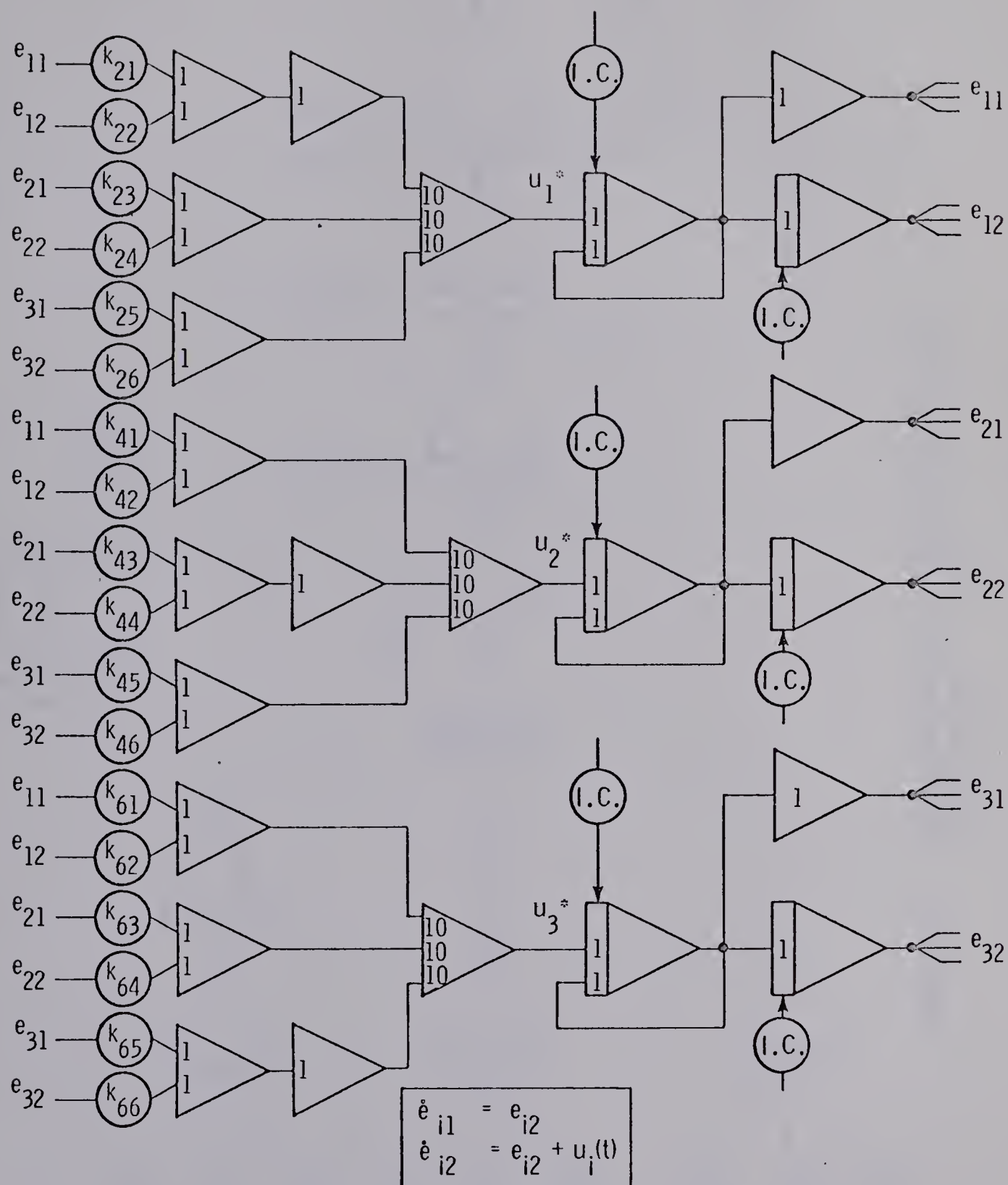


Figure 4-1 ANALOGUE SIMULATION OF SECOND-ORDER SYSTEM

forces is most apparent.

3. Catastrophic failure. A negative ramp change in velocity error of train #1 was superimposed on the system to simulate the first train decelerating at a high and uncontrolled rate.

Before discussing the results of the above tests it may be useful to indicate the relationships between the second and third-order control variables and the acceleration and jerk forces.

For the second-order system, from 2.25

$$u_i(t) = \frac{f_i(t) - d_0}{b_i} \quad (4.1)$$

where d_0 is the drag force on the i th train when there is no velocity error and b_i is a constant of proportionality.

By differentiating 4.1 with respect to time we get

$$\dot{u}_i(t) = \frac{\dot{f}_i(t)}{b_i} \quad (4.2)$$

which relates the control variable to jerk forces for the second-order system.

For the third-order system, from 2.24

$$w_i(t) = \frac{\dot{f}_i(t)}{c_i} \quad (4.3)$$

and integrating the above yields

$$\int_0^{\infty} w_i(\tau) d\tau = \frac{f_i(t)}{c_i} \quad (4.4)$$

which relates the accelerating force to the control variable for the third-order system.

4.2 Results of Test 1

In the following, we will compare the overall behaviour of the second and third-order control systems as well as the effect of the different cost functionals on the two systems.

The ratio of $e_{11}(0)/e_{12}(0)$ was chosen arbitrarily to be 10/3 for all the following tests. Referring to section 3.2 it can be seen that cost functionals J_1 , J_3 and J_4 are of the same form but have different weighting on the term $\sum_{i=1}^2 (e_{i1} - e_{i+1,1})^2$ which penalizes for error in headway from schedule. Second-order system response to test 1 using these cost functionals is shown in figures 4-3(J_1), 4-4(J_3), and 4-7(J_4). The corresponding third-order system response is shown in figures 4-5 and 4-6(J_3) and 4-10(J_1 , J_3 and J_4). It should first be noted that the second-order system response for cost J_3 agrees with that obtained by Levine and Athans which is some assurance that our simulation is correct.

First, it may be useful to note the similarities in response of the second and third-order systems using these cost functionals.

1. As predicted in section 2.4, the cost functionals J_1 , J_3 and J_4 do not guarantee a zero steady state position error for any of the three trains. However, the steady state value of

position error for all trains is equal, constant and less than the value at time $t=0$. The actual value at steady state varies depending on the cost functional used (relative magnitudes are indicated in the figures).

2. For both systems, with increased weighting on the term $\sum_{i=1}^2 (e_{i1} - e_{i+1,1})^2$ in the cost functional, the attempted correction in position error of a given train takes place in less time and is accompanied by a greater amount of overshoot in the state variables e_{i1} and e_{i2} . Excessive overshoot of any state variable is undesirable from the standpoint of passenger comfort. Approximate transient time intervals for each system are indicated on the figures.

Next, it is possible to point out the differences in behaviour of the second and third-order systems for equivalent cost functionals.

1. Generally, the transient intervals for the second-order systems are less than those for the corresponding third-order systems. For instance, in the case of cost J_3 , the second-order system (figure 4-4) exhibits about 5 seconds of transient behaviour while the third-order system (figure 4-5) exhibits a transient time of about 9 seconds.
2. The damping of the second-order system for a given cost functional appears to be greater than that for the corresponding third-order system. For example, considering the systems with cost J_3 (figures 4-4 and 4-5), the state $e_{11}(t)$ clearly shows less overshoot in the case of the second-order system than it does in the third-order system.

3. The value of steady state position error for the trains, given any one of the three cost functionals considered, is less for the third-order system than for the second-order system.

Let us now consider the systems corresponding to the cost functionals J_6 and J_7 . Referring to figures 4-8(J_6) and 4-9(J_7) for the second-order system and to figure 4-10 for the third-order system, the following comments can be made.

1. Cost functional J_6 is identical to J_3 except for the added term $10 \sum_{i=1}^2 (e_{i2} - e_{i+1,2})^2$ which penalizes the system for differences in velocity errors between successive trains.

As was predicted in section 2.4, the addition of this term does not appreciably affect the behaviour of either the second or third-order systems compared to that using cost J_3 . Comparing the second and third-order systems using cost J_6 , one can make the same observations as were made concerning the systems using cost J_1 , J_3 and J_4 .

2. Cost functional J_7 penalizes the system only for differences in velocity errors between successive trains and as can be seen in figure 4-9 (second-order system) the steady state position errors are different for each train in the string. This is to be expected since no penalty is imposed for differences in position errors between successive trains as was the case for cost functionals J_1 , J_3 , J_4 , and J_6 . Hence this controller does not regulate for headway and cannot be considered satisfactory in meeting our objective. Although only the state $e_{11}(t)$ is shown for J_7 for the third-order system in figure 4-10, system behaviour is very similar to that shown

in figure 4-9 for the second-order system.

Lastly, let us consider the systems corresponding to cost J_9 which differs from cost J_3 in that the term $10 \sum_{i=1}^3 e_{i1}^2$ is added inside the integral. This has a marked effect on the system behaviour to test 1 in that steady state position errors for all trains are zero. Referring to figures 4-11 (second-order) and 4-12 (third-order) the following can be stated.

1. Since steady state position errors are zero for all trains, this controller regulates for headway and scheduled position. Also, for both the second and third-order systems the initial condition of train #1 going too fast does not disturb the following trains to a great extent during the correction interval. This is of course due to the fact that all position errors remain at zero for the following trains once the transient interval has passed.
2. Comparing the two systems, one can easily see that the transient period is greater for the third-order system. Both systems display some overshoot which could be eliminated by reducing the weighting on both error terms in the cost functional J_9 . Generally, this cost functional results in the controller best meeting our objectives for both systems. As a result of test 1, we cannot say which of the two systems is to be preferred.

4.3 Results of Test 2

To simulate a negative step change in velocity error (e_{12}) of the first train, a negative step voltage change in e_{12} was automatically

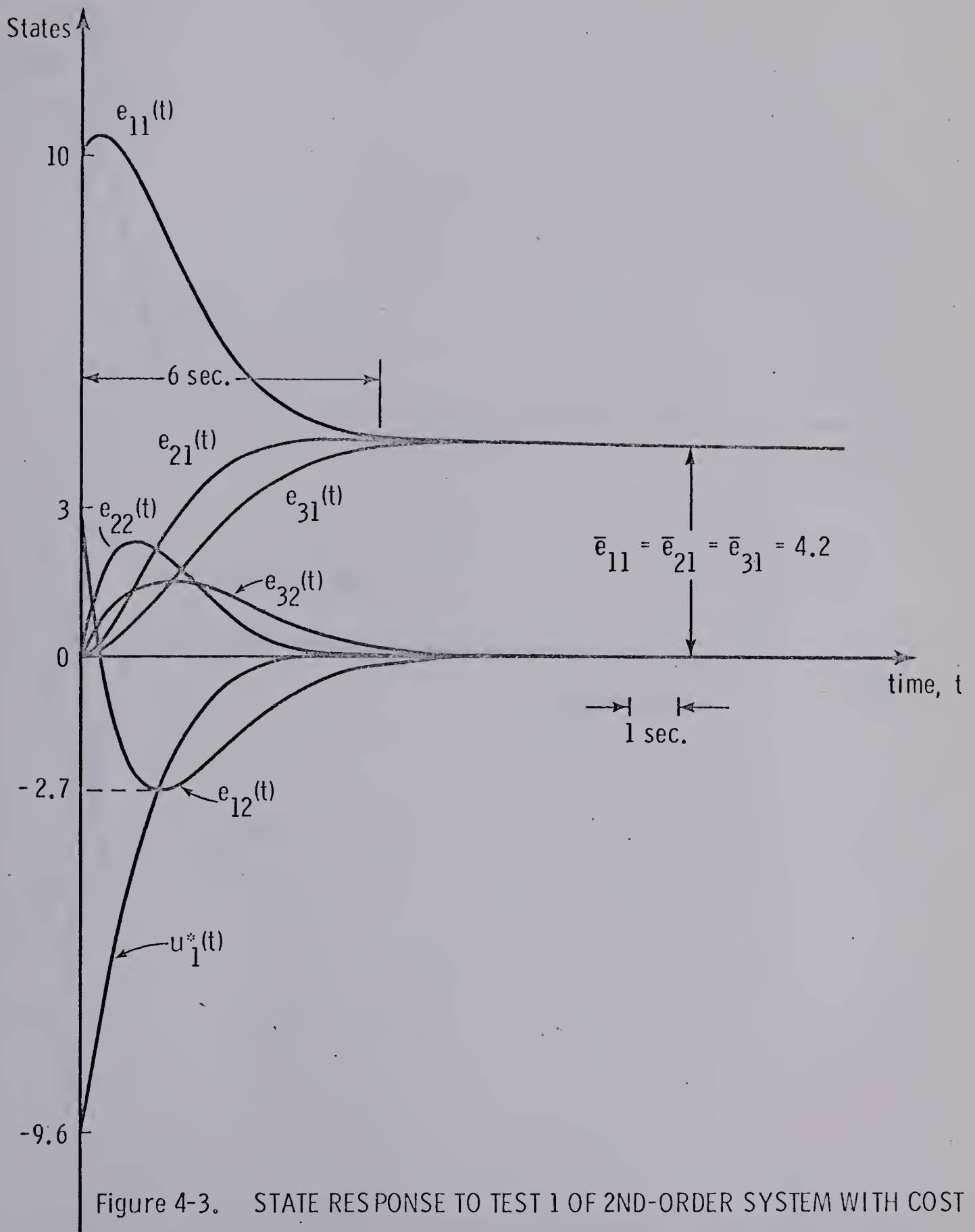


Figure 4-3. STATE RESPONSE TO TEST 1 OF 2ND-ORDER SYSTEM WITH COST J_1

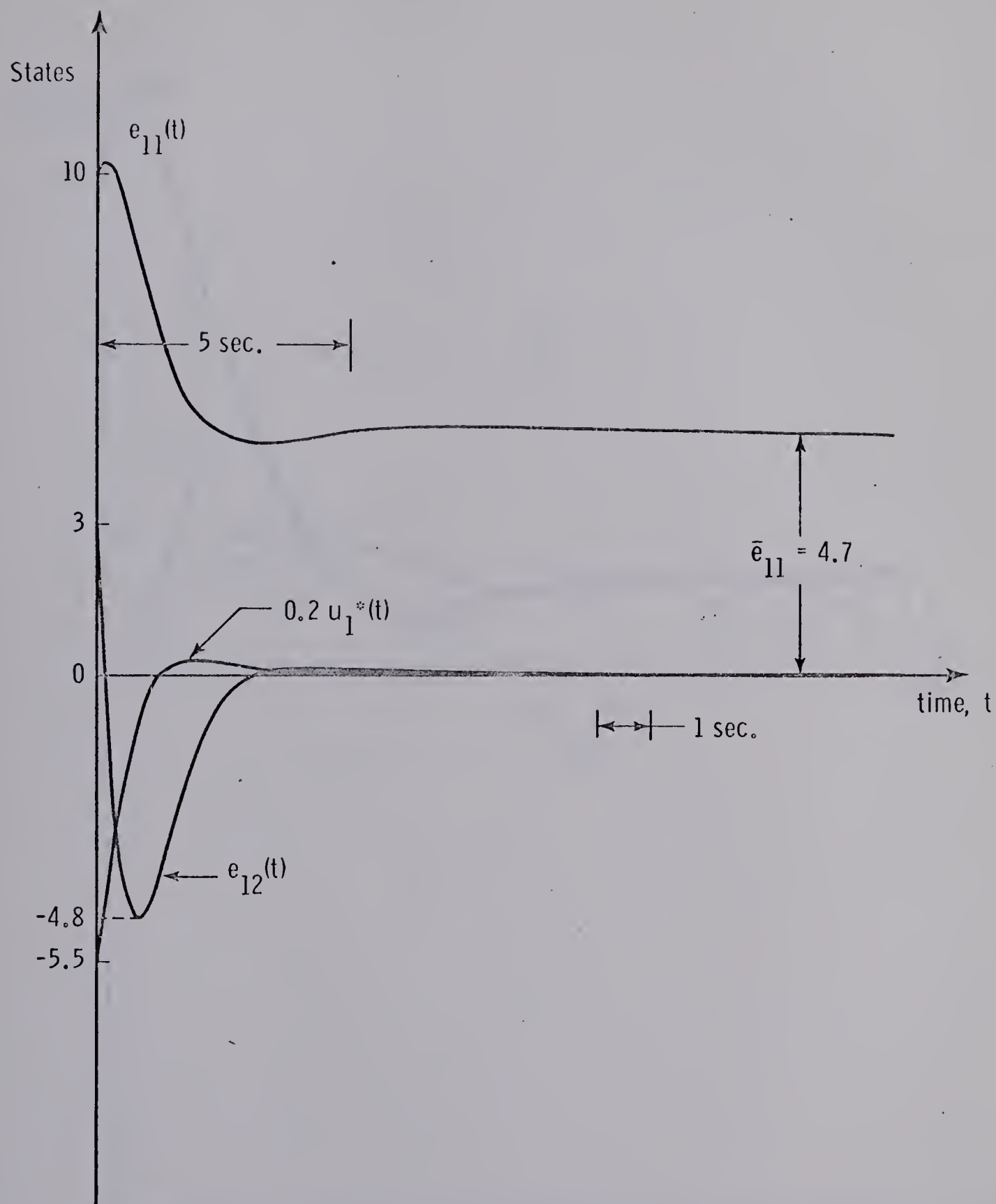


Figure 4-4 STATE RESPONSE TO TEST 1 OF 2ND-ORDER SYSTEM WITH COST J_3

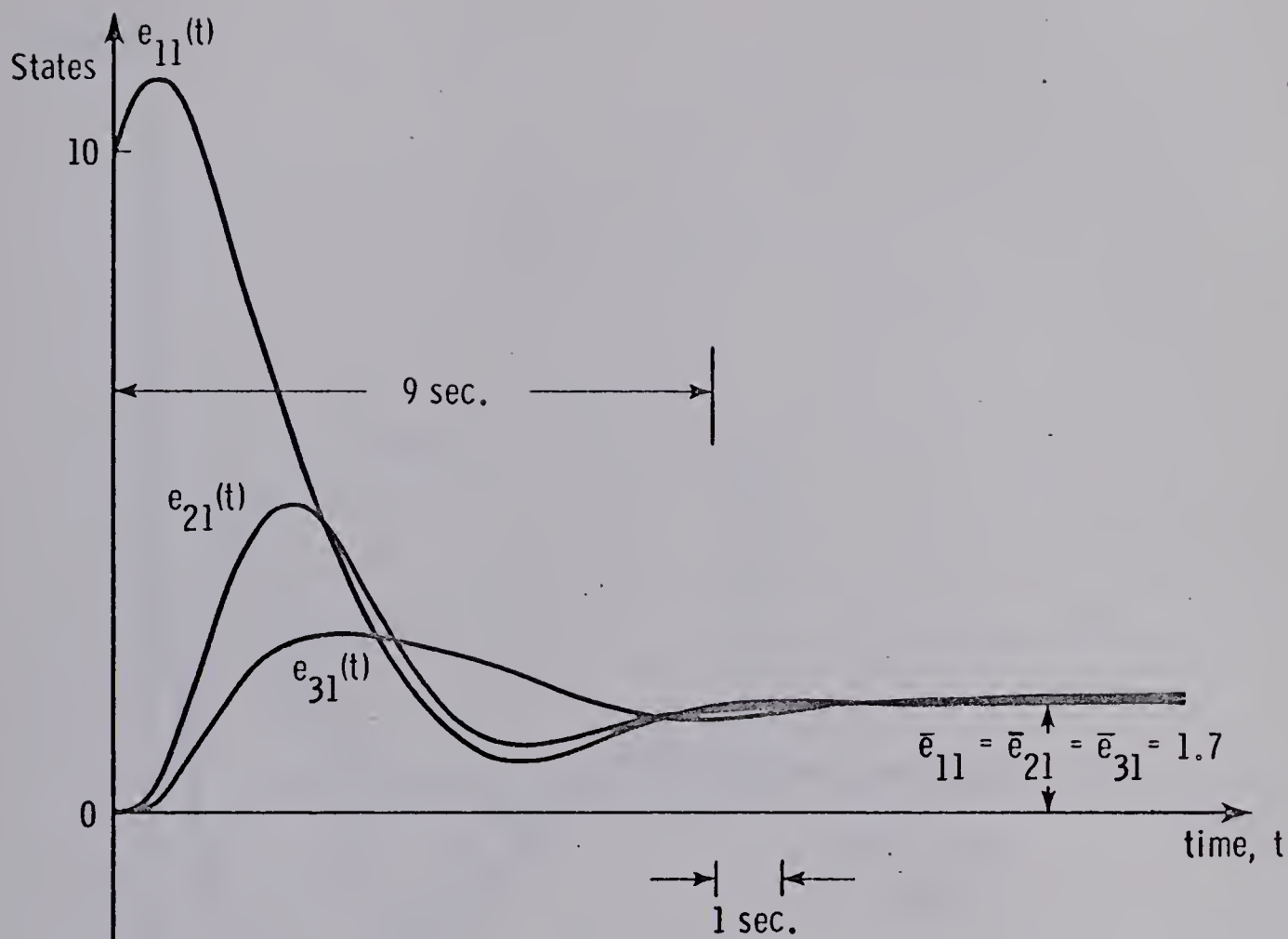


Figure 4-5 POSITION STATE RESPONSE TO TEST 1 OF 3RD-ORDER SYSTEM WITH COST J_3

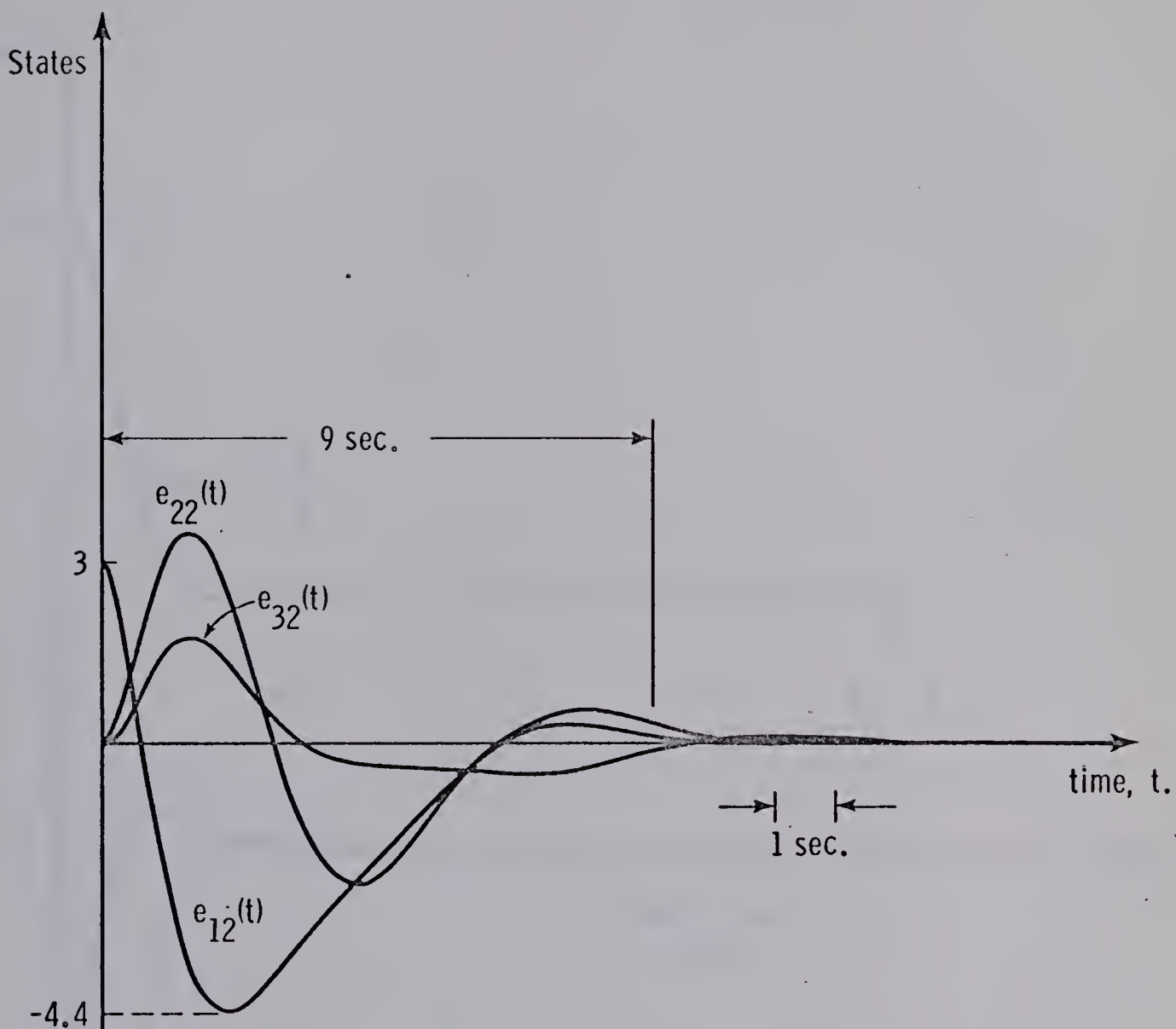
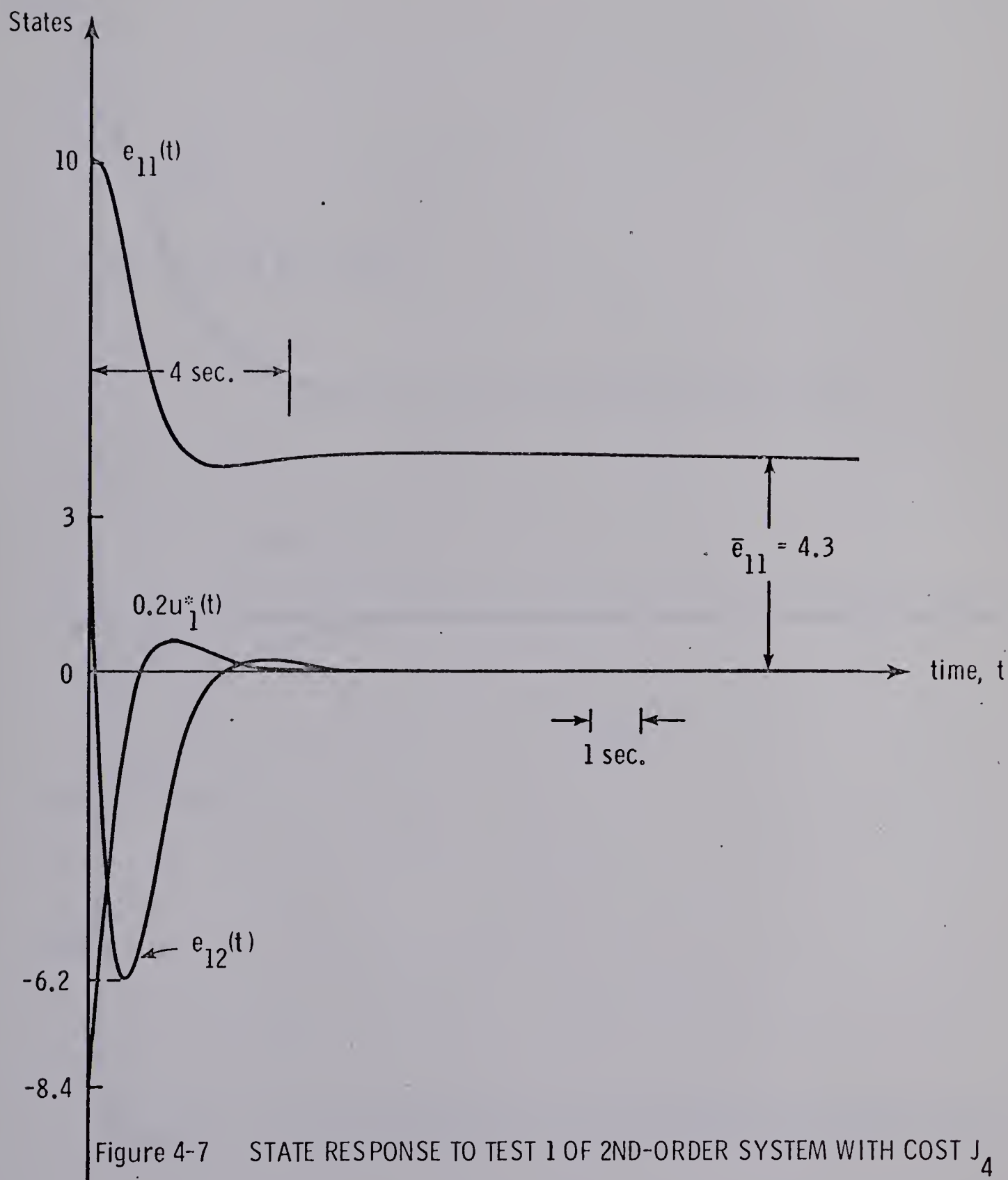


Figure 4-6 VELOCITY STATE RESPONSE TO TEST 1 OF 3RD-ORDER
SYSTEM WITH COST J_3



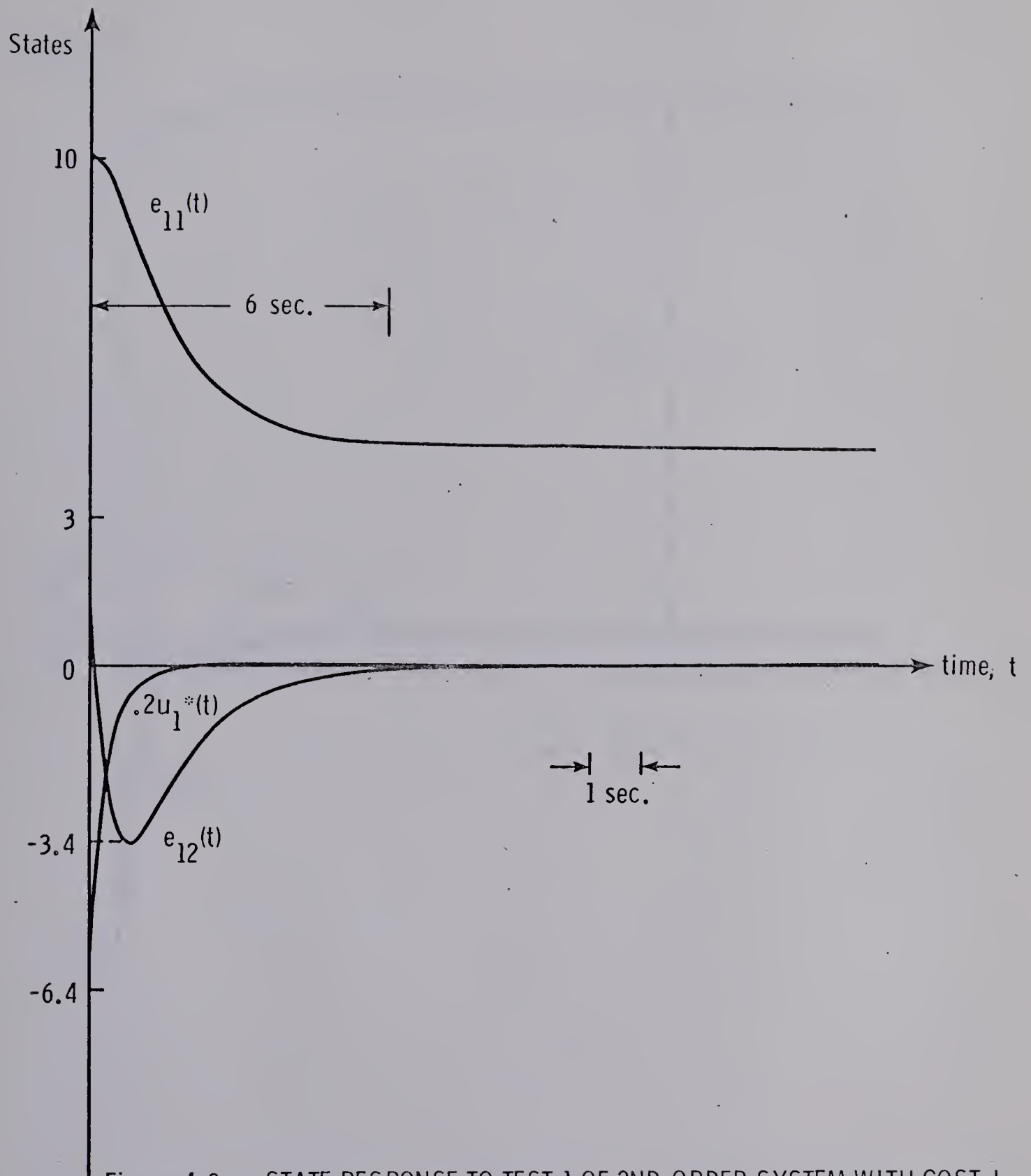


Figure 4-8 STATE RESPONSE TO TEST 1 OF 2ND-ORDER SYSTEM WITH COST J_6

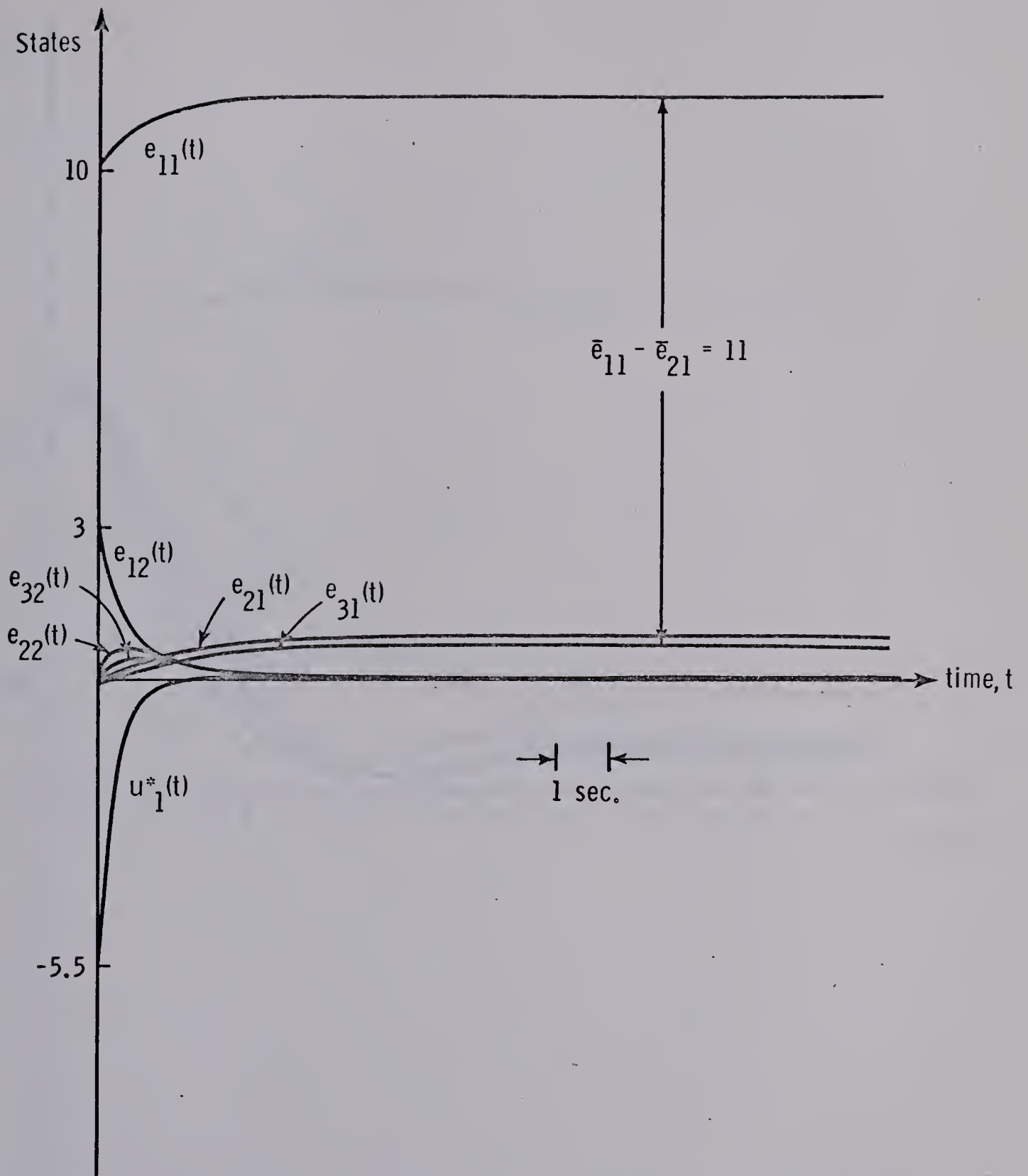


Figure 4-9 STATE RESPONSE TO TEST 1 OF 2ND-ORDER SYSTEM WITH COST J_7

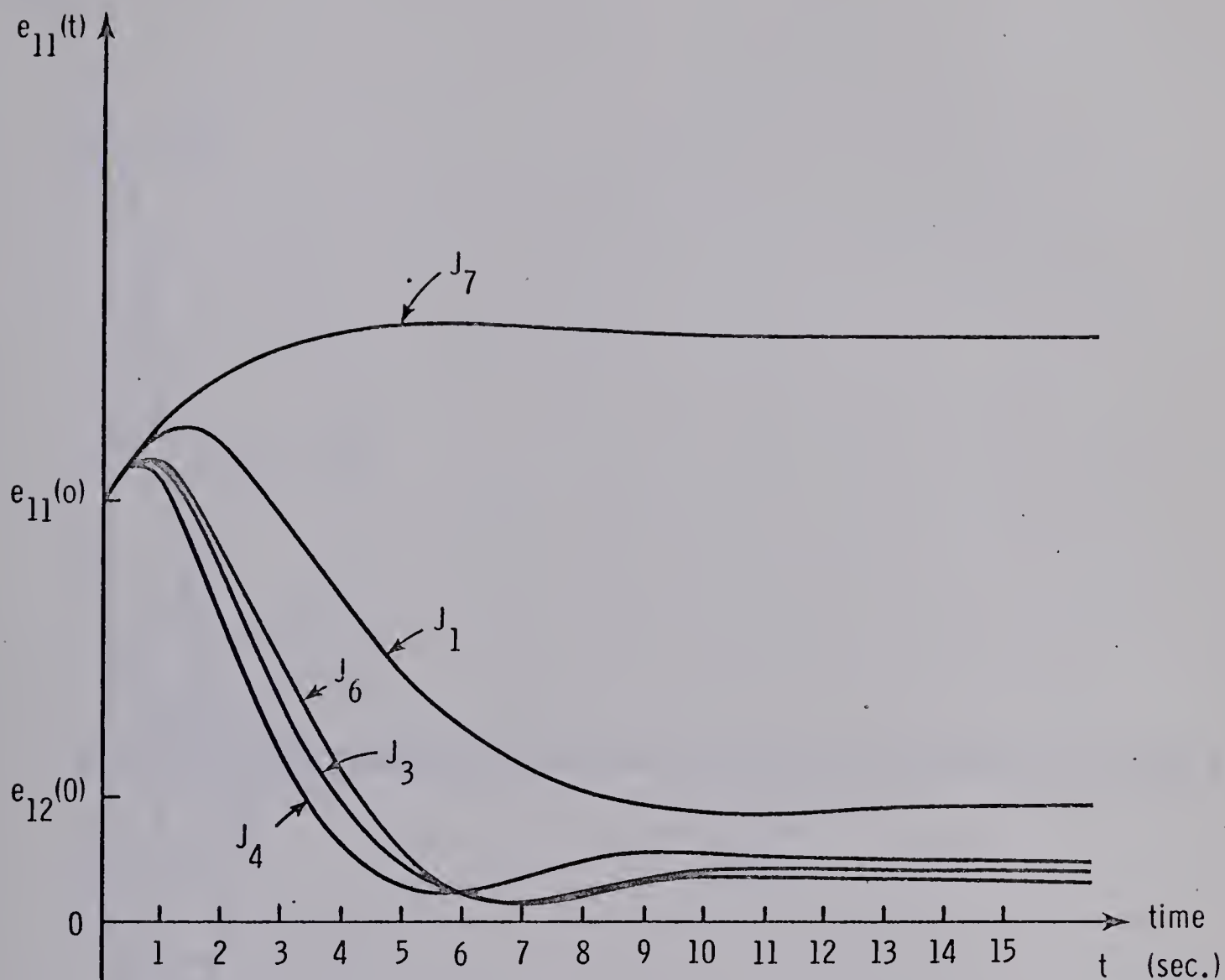


Figure 4-10 POSITION ERROR OF TRAIN 1 VS. TIME FOR THIRD-ORDER SYSTEM WITH VARIOUS COST FUNCTIONALS.

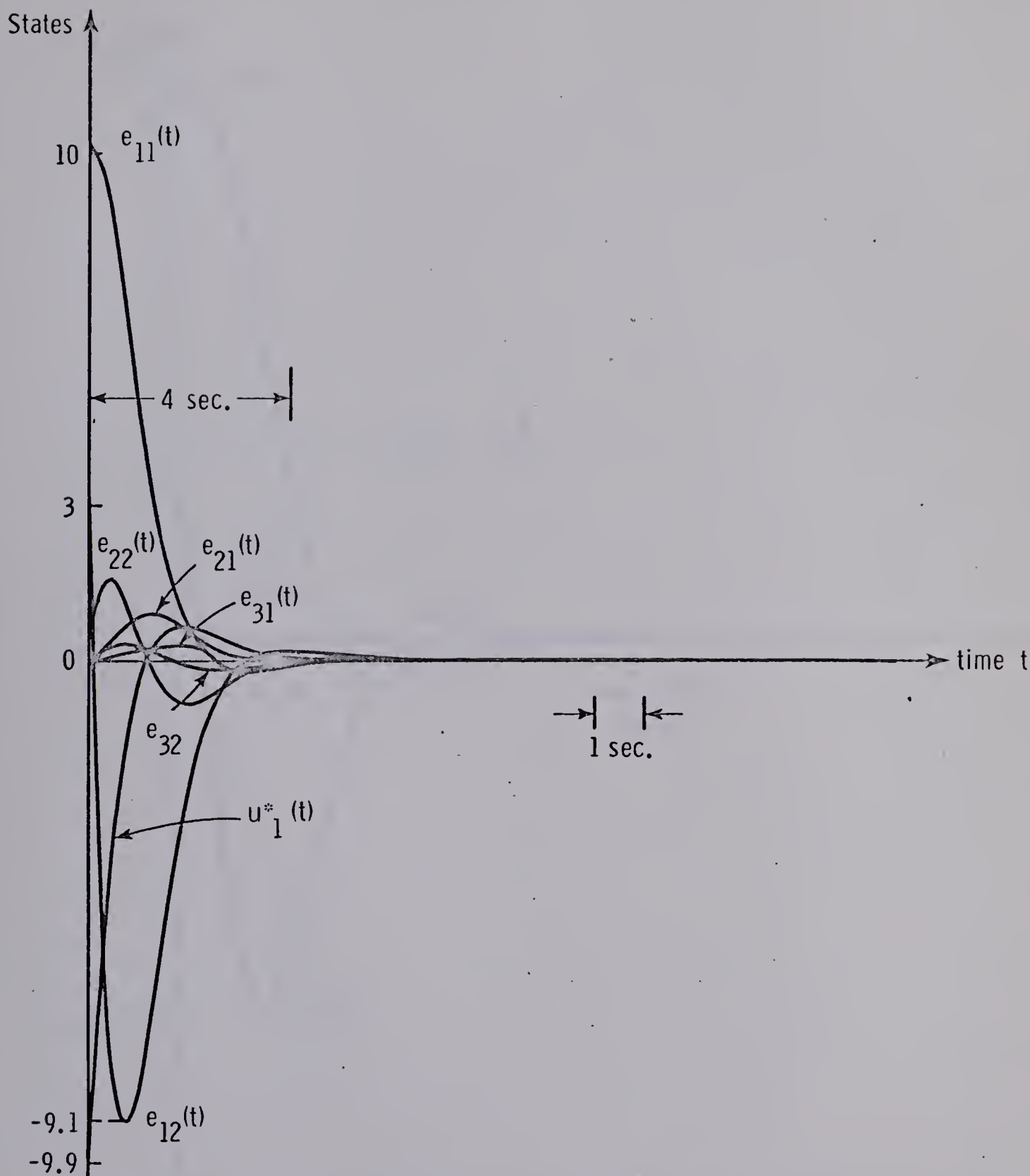


Figure 4-11 STATE RESPONSE TO TEST 1 OF 2ND-ORDER SYSTEM WITH COST J_9

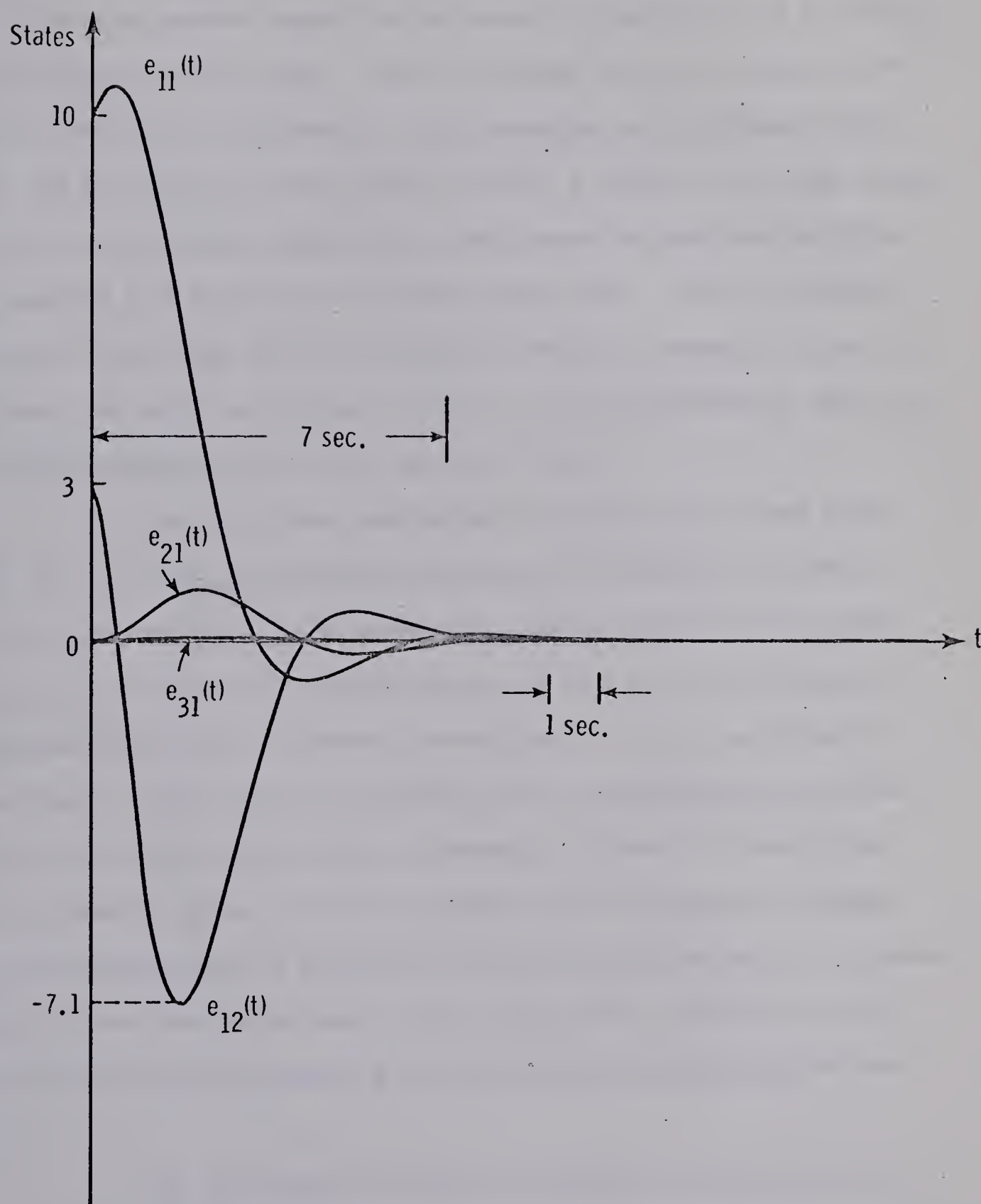


Figure 4-12 STATE RESPONSE TO TEST 1 OF 3RD-ORDER SYSTEM WITH COST J_9

switched into the circuit after 3 seconds of unperturbed operation.

Cost functional J_3 was used in this test. Figures 4-13a and 4-13b show the second-order system control variable and its derivative as a function of time (for the first train). The step change in $e_{12}(t)$ occurs at $t=3$ seconds at which time the control $u_1(t)$ undergoes a step change (figure 4-13a) and then decays to zero again in about 4 seconds. The step change in $u_1(t)$ at $t=3$ seconds results in a step change in accelerating force (see equation 4.1) and hence an infinite jerk force. This is verified in figure 4-13b though the pen recorder is unable to exactly follow the response. The noise on this plot is due to the differentiator used and was reduced somewhat by use of a low-pass filter.

The third-order system was subjected to the same step change in e_{12} as the second-order system and the results are shown in figures 4-13c and 4-13d. The third-order system control for the first train, $w_1(t)$, behaves in a similar manner as does $u_1(t)$ in the case of the second-order system. However, since from 4.3, $w_1(t)$ is directly proportional to jerk force, no infinite jerk is experienced as was the case for the second-order system. Comparing figures 4-13d and 4-13a the accelerating forces of the two systems can be compared. Although the accelerating force is applied to the third-order system for a greater length of time than in the case of the second-order system, the application of the force is made in a smoother manner, eliminating infinite jerk.

The differences in the two systems indicated above will be evident for any sudden change in one of the states since the accelerating force for the third-order system is the integral of the control

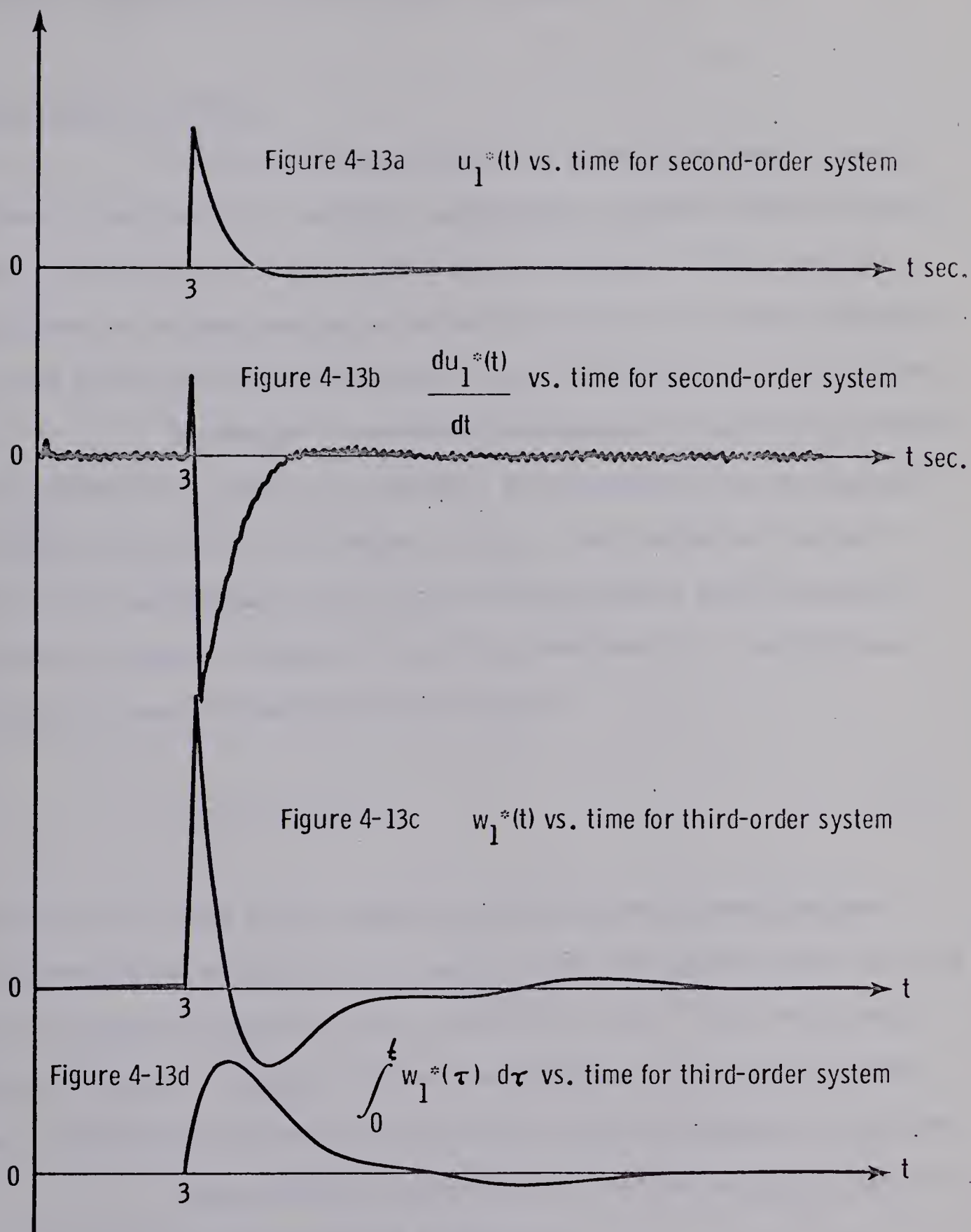


FIGURE 4-13 RESULTS OF TEST 2 (COST J_3)

variable while for the second-order system the accelerating force is directly proportional to the control variable.

4.4 Results of Test 3

In the previous two tests no attempt was made to assign values to the position, velocity, acceleration and jerk curves although this could be done. In the third test, in which the first train was simulated to be decelerating at an uncontrolled rate, we have attempted to make some quantitative statements about the behaviour of the systems.

To simulate a constant deceleration of train #1 the feedback paths for e_{11} and e_{12} were opened and a negative ramp voltage substituted for e_{12} and its integral for e_{11} . The results of the test using cost functionals J_8 and J_3 for both the second and third-order systems are shown in figures 4-14, 4-15, 4-16 and 4-17. By differentiating the second equation of 2.13 we get

$$\dot{e}_{12}(t) = \dot{x}_{12}(t) \quad (4.5)$$

and hence the slope of the imposed $e_{12}(t)$ ramp can be assigned some deceleration value which will be equal to the deceleration rate of train #1. For this test the value of 1.0 g. (or 32.2 ft./sec.²) was arbitrarily given to $\dot{e}_{12}(t)$. Once this value is established, the position states can be calibrated in feet and the velocity states calibrated in ft./sec.

Under constant deceleration, a collision between trains 1 and 2 will occur when for the first time

$$|e_{11} - e_{21}| = h_0 = kx_0, \quad k > 0 \quad (4.6)$$

Figures 4-18 (second-order) and 4-20 (third-order) show the relationship between $|e_{11} - e_{21}|$ and time for both cost functionals J_3 and J_8 . From 4.6, at collision, $|e_{11} - e_{21}| = h_0$, the headway, so these figures can be thought of as plots of headway vs time before collision between trains 1 and 2. For a given collision time we can then determine the velocity of impact given by

$$v_{\text{impact}} = e_{22} - e_{12} \quad (4.7)$$

Figures 4-19 and 4-21 are plots of headway vs v_{impact} for the second and third-order systems respectively. The following comments on test 3 can now be made.

1. For both the second and third-order systems, cost J_3 resulted in better maintenance of spacing between the 1st and 2nd trains than did cost J_8 (as evidenced in figures 4-18 and 4-20). This is due to the penalty that cost J_8 imposes on the system for large velocity errors. At any time, t , the velocity error of train 2 with cost J_8 is less than that with cost J_3 which means that the system with cost J_8 is unable to make corrections in the position of the 2nd train as rapidly as the system with cost J_3 .
2. In comparing second and third-order system behaviour, it should be noted that in the case of the second-order system with both cost functionals J_8 and J_3 , for large enough headway the impact velocity approaches a constant (figure 4-19). In other words, there is a small delay on the part of the 2nd train in reaching the deceleration rate of the 1st train and

thus for large headways (corresponding to high cruising velocities) the delay causes an initial offset in $(e_{22} - e_{12})$ after which both trains decelerate at the same rate.

For the third-order system with cost J_8 , the deceleration rate of the 2nd train never becomes as great as that of the 1st train and hence the impact velocity is proportional to headway for large enough headway (figure 4-21).

For the third-order system with cost J_3 , the 2nd train decelerates at a greater rate than the 1st train thereby overcoming the time delay effect. After about 13 sec. (figure 4-17) both trains decelerate at the same rate. As can be seen in figure 4-21 there will be no collisions for headways greater than 300 feet (for a deceleration rate of 1.0 g.). Due to time limitations, this unusual behaviour of the third-order system was not investigated further.

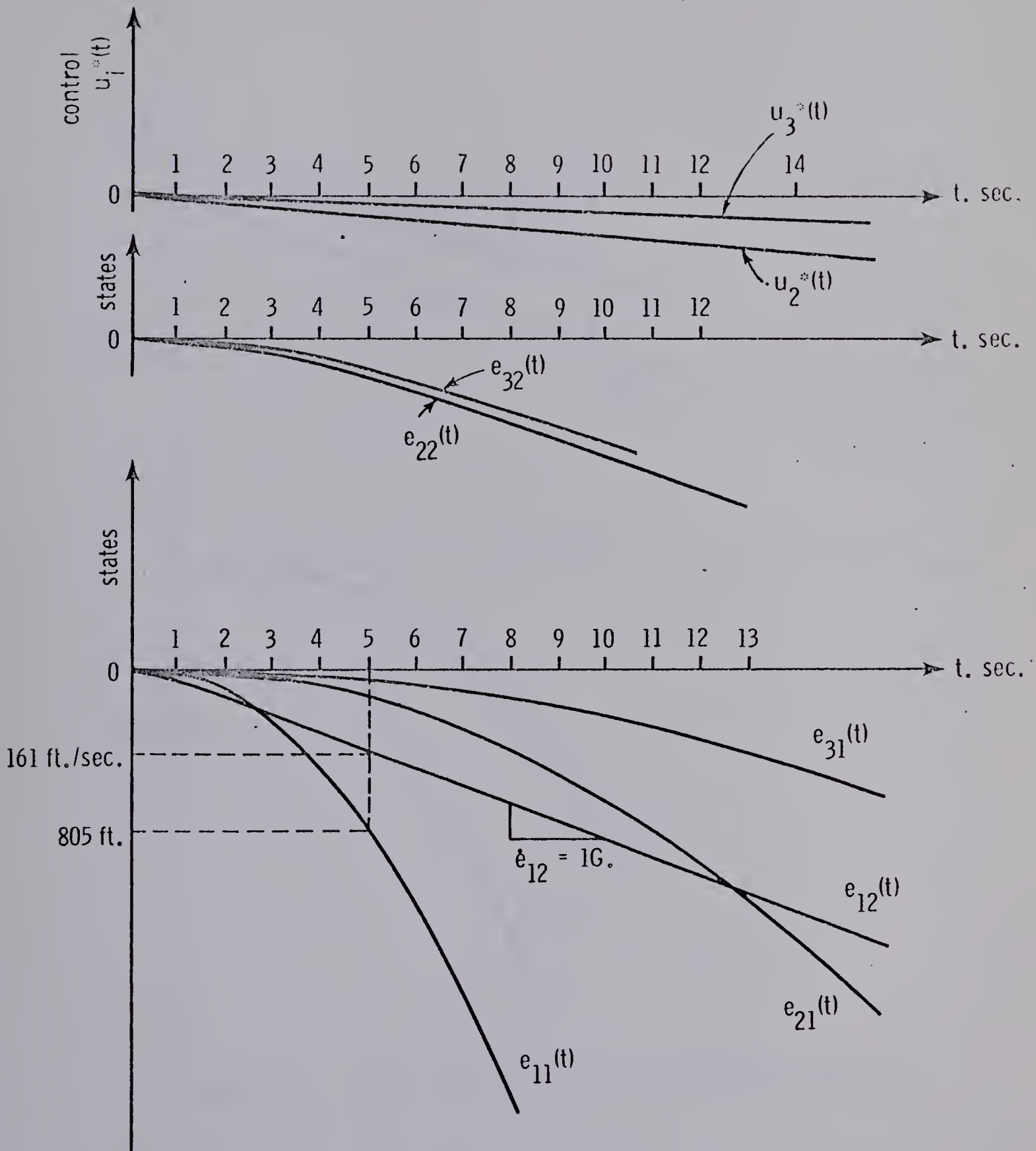


Figure 4-14 RESULTS OF TEST 3 FOR 2ND-ORDER SYSTEM WITH COST J_8

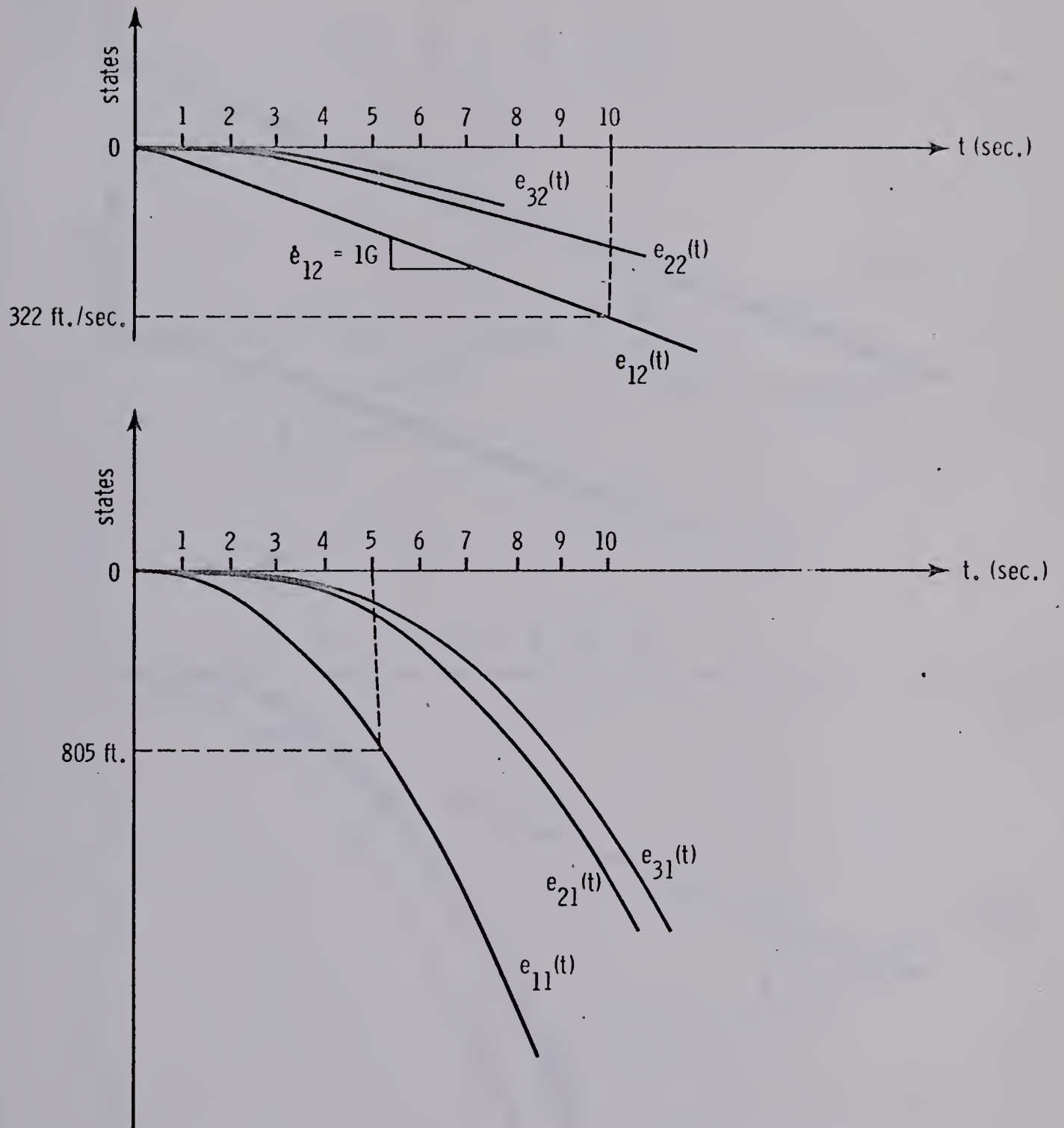


Figure 4-15

RESULTS OF TEST 3 FOR 3RD-ORDER SYSTEM WITH COST J_8

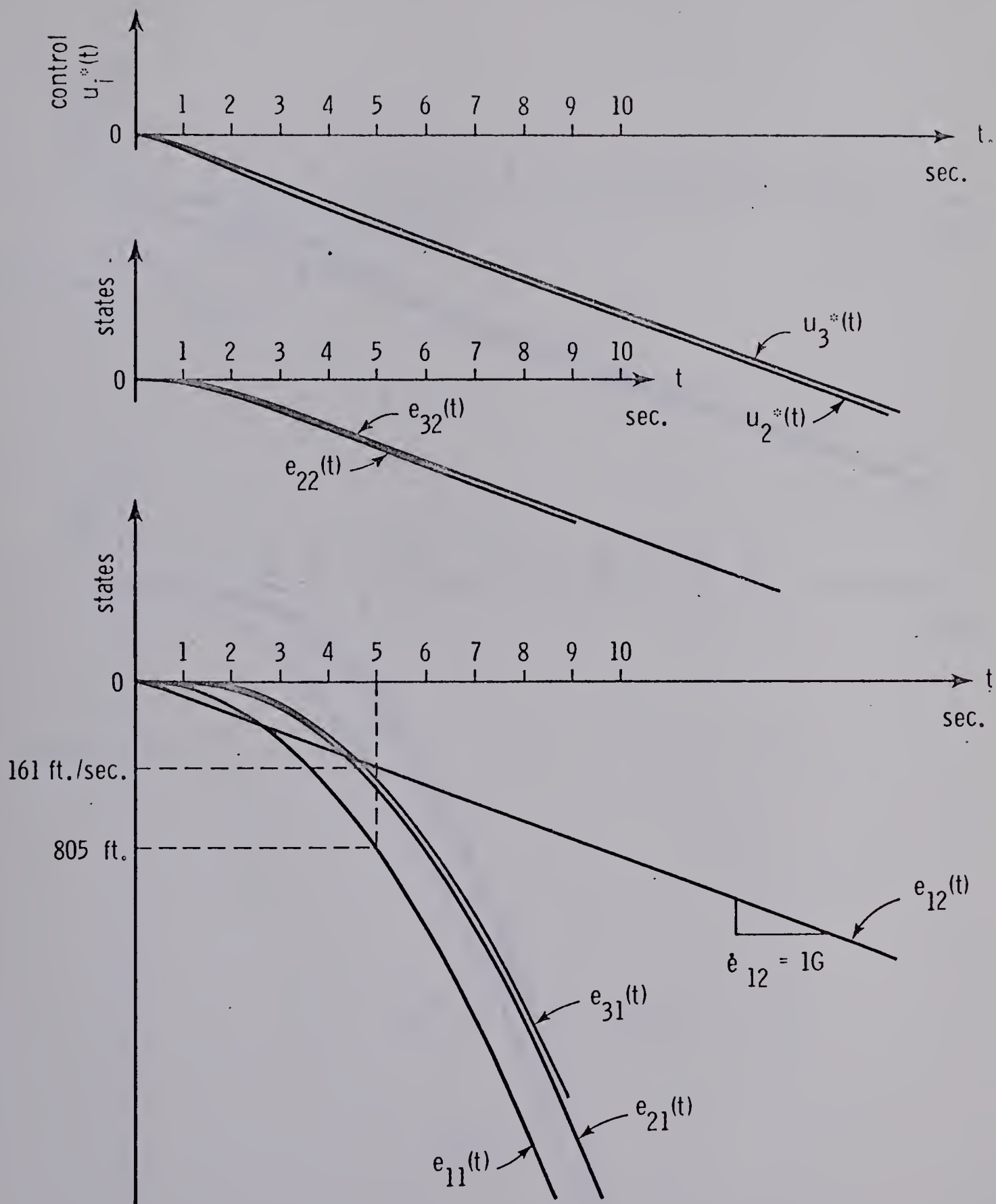


Figure 4-16 RESULTS OF TEST 3 FOR 2ND-ORDER SYSTEM WITH COST J_3

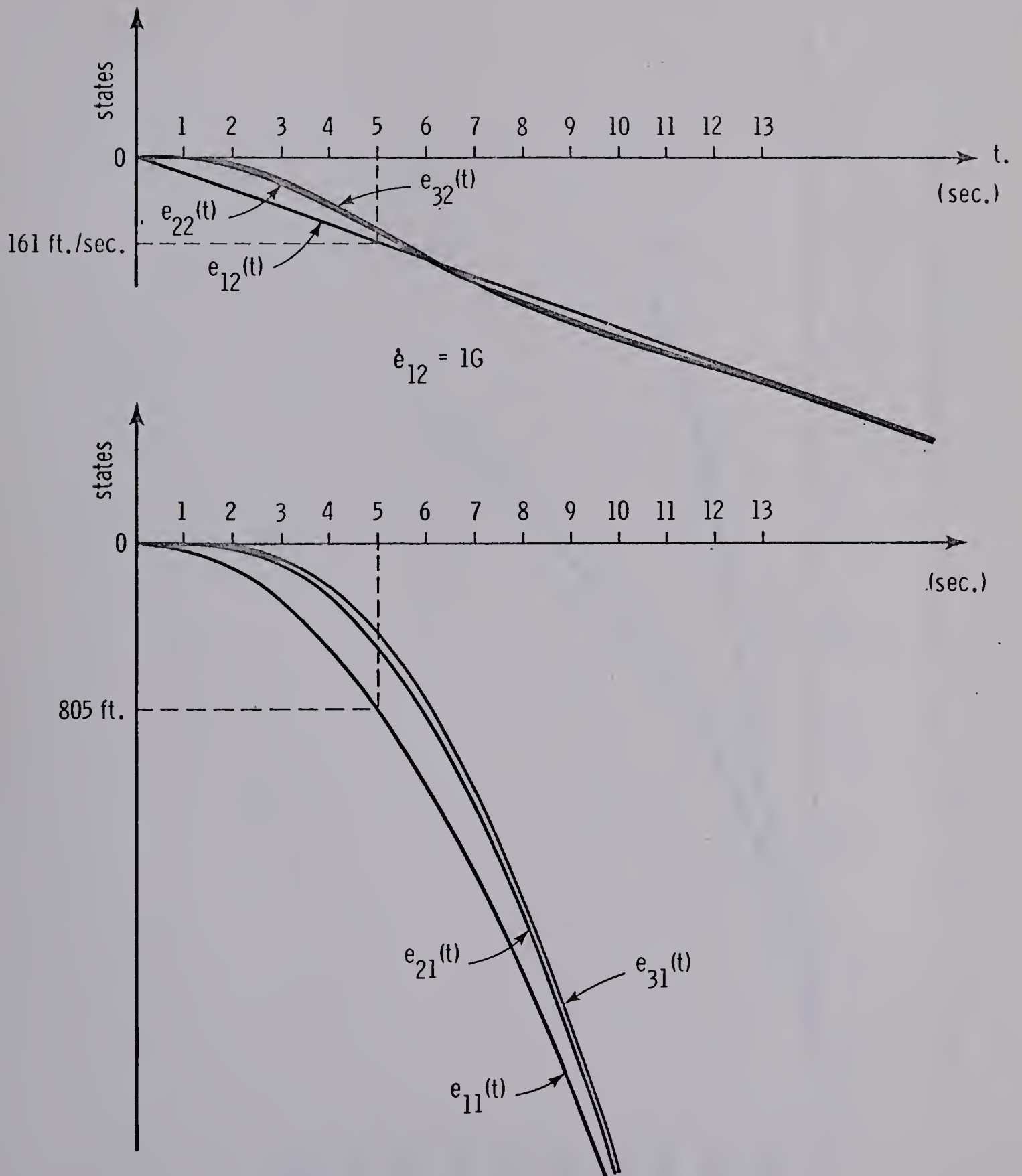


Figure 4-17 RESULTS OF TEST 3 FOR 3RD-ORDER SYSTEM WITH COST J_3

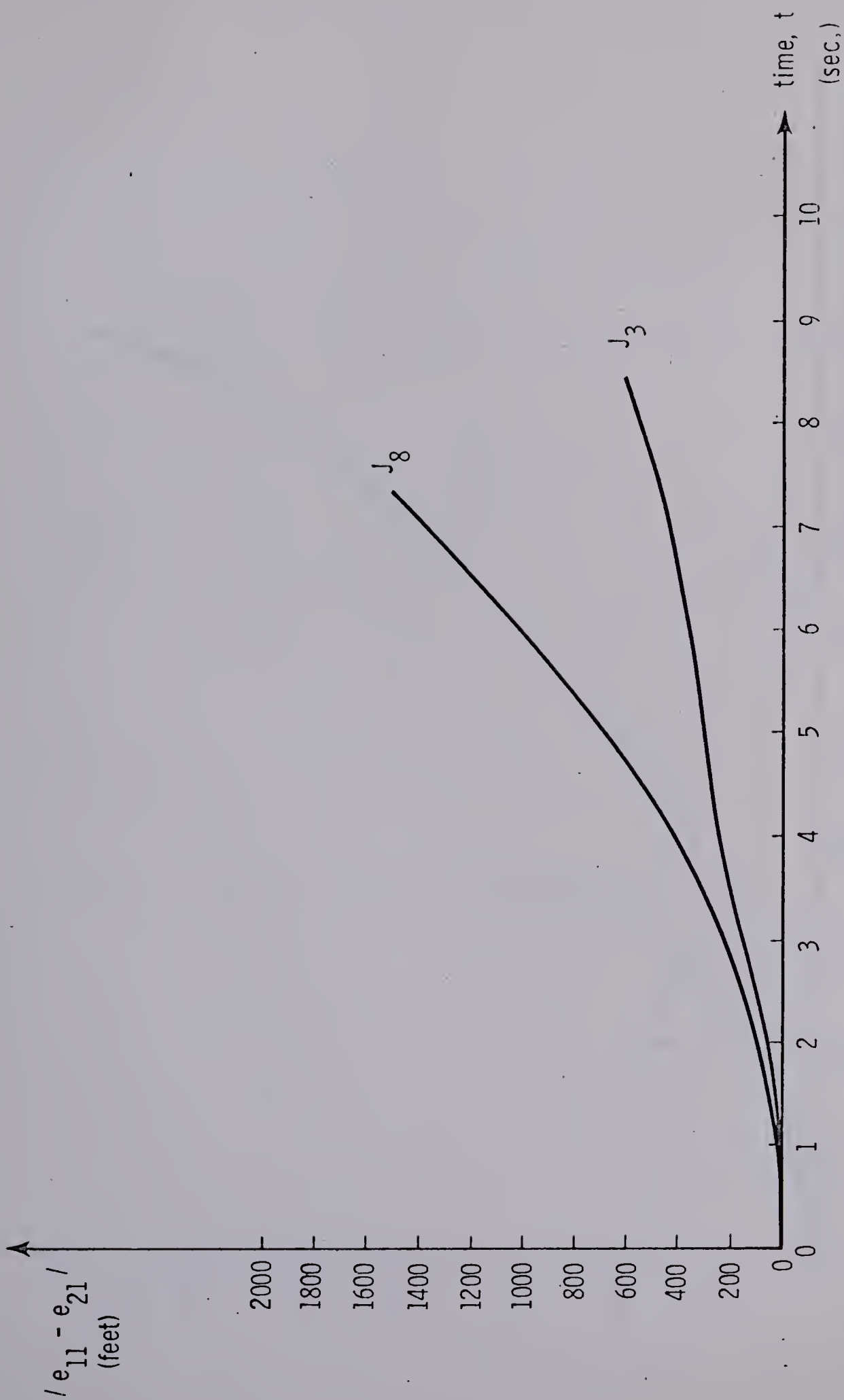


Figure 4-18 $|e_{11} - e_{21}|$ VS. TIME FOR SECOND-ORDER SYSTEM SUBJECT TO TEST 3

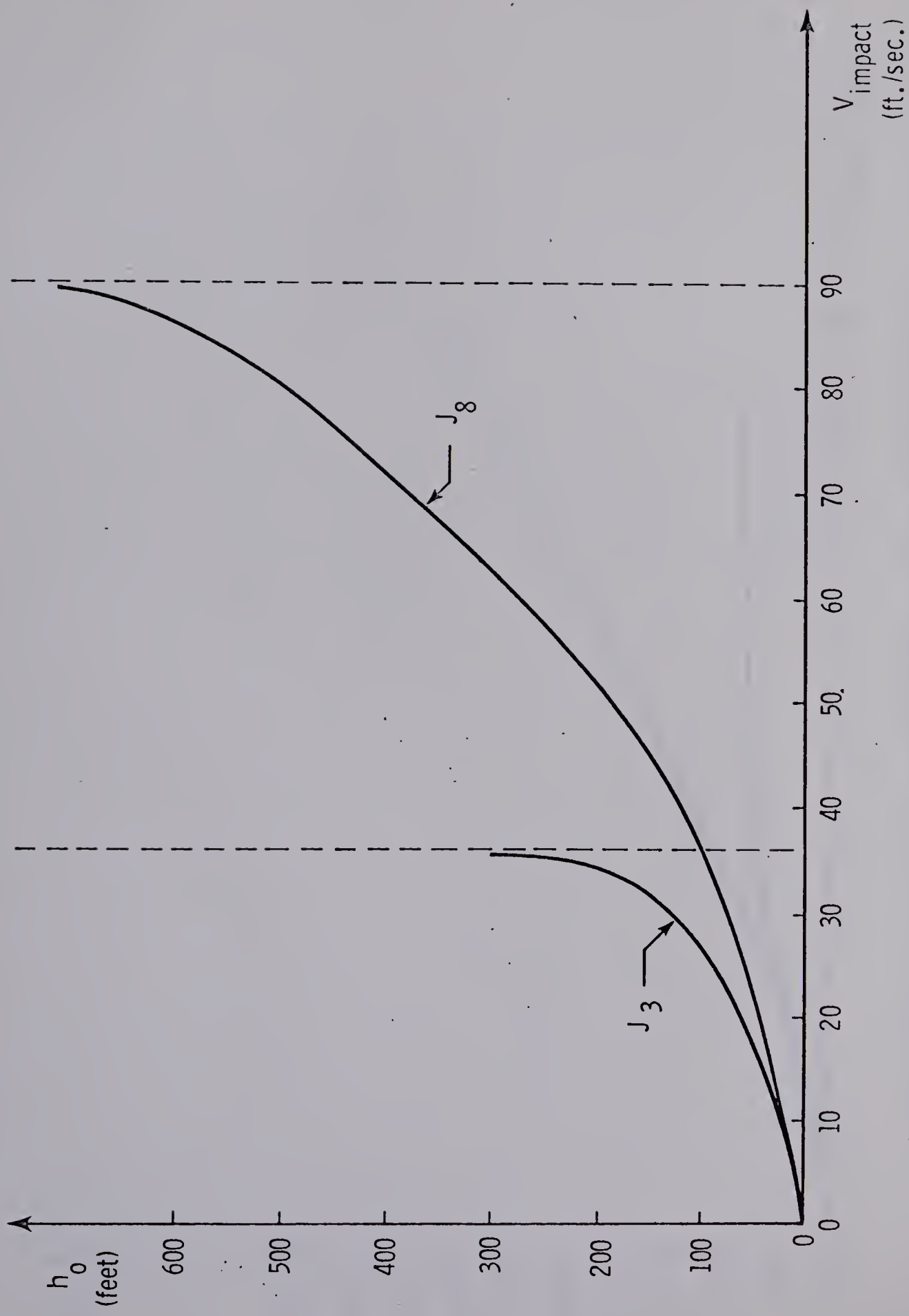


Figure 4-19 HEADWAY VS. VELOCITY OF IMPACT FOR THE SECOND-ORDER SYSTEM

SUBJECT TO TEST 3

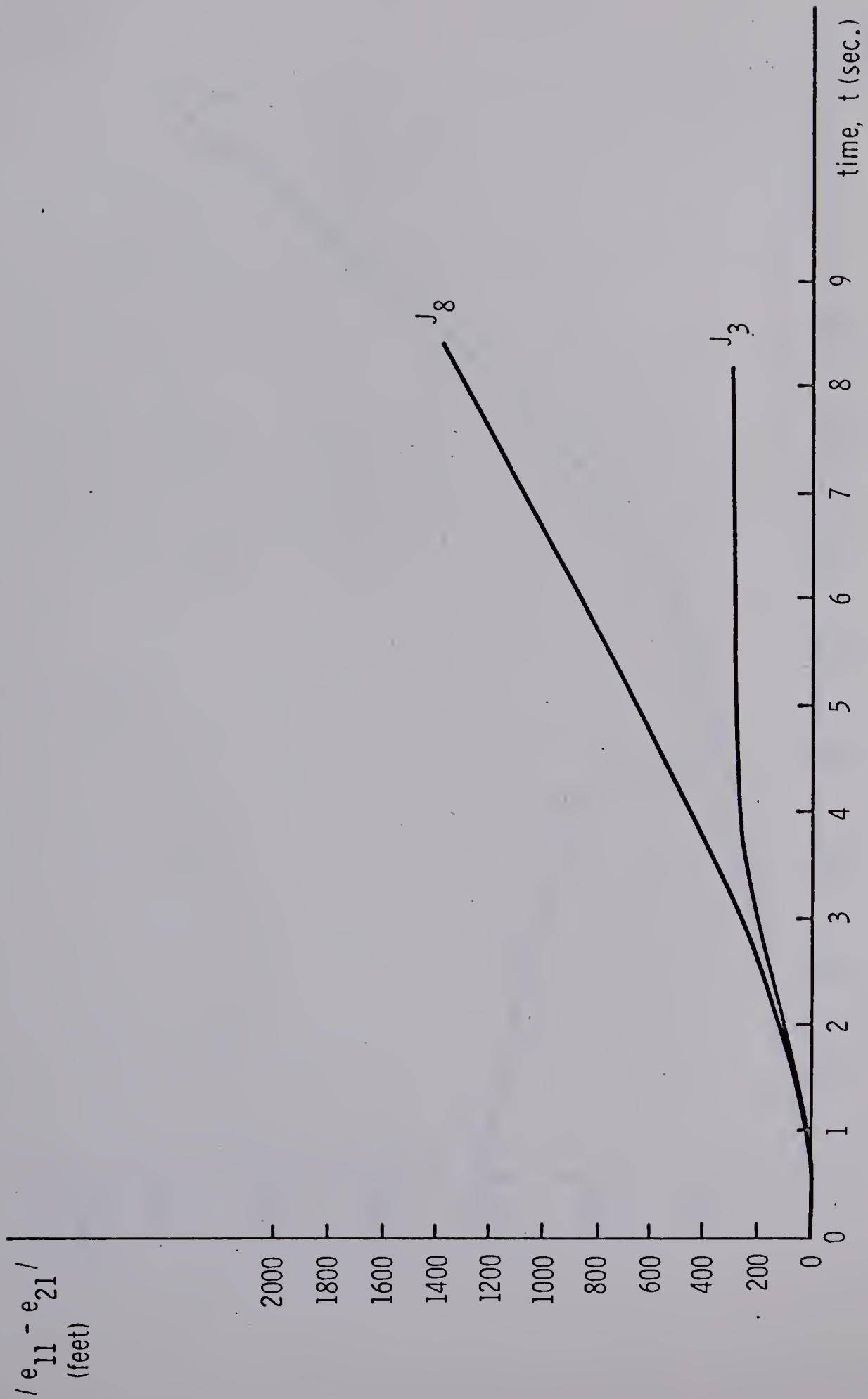


Figure 4-20 $/e_{11} - e_{21}/$ VS. TIME FOR THIRD-ORDER SYSTEM SUBJECT TO TEST 3

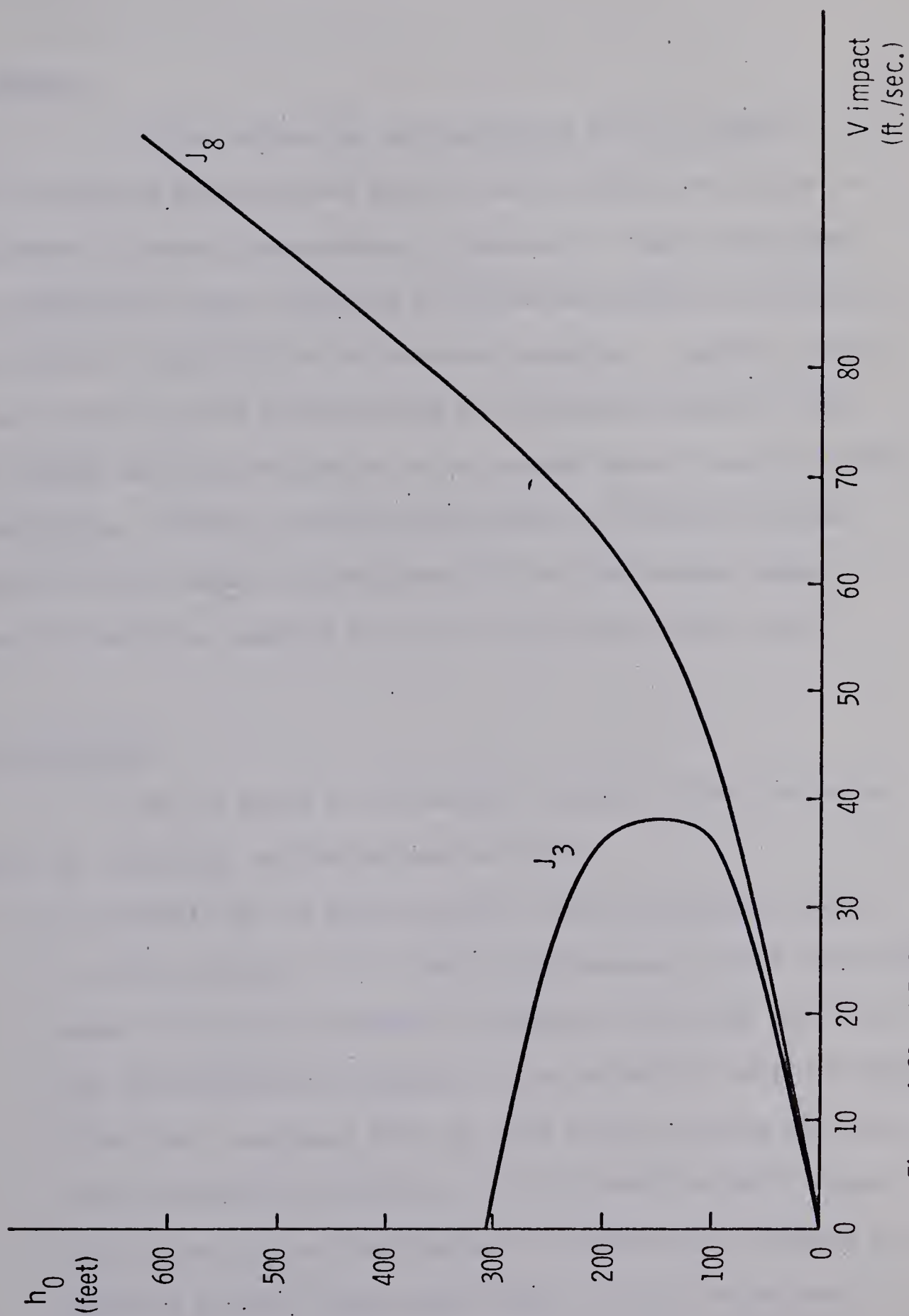


Figure 4-21 HEADWAY VS. VELOCITY OF IMPACT FOR THE THIRD-ORDER SYSTEM
SUBJECT TO TEST 3

CHAPTER 5

SUMMARY, CONCLUSIONS AND SUGGESTIONS

FOR FUTURE RESEARCH

5.1 Summary

Before making any conclusions it will be useful to briefly summarize what has been done in this thesis in so far as the development of optimal controllers is concerned. First, the second-order mathematical model developed by Levine and Athans was studied in some detail by simulation on the analogue computer. A greater number of cost functionals was utilized than in the previous study of this model thereby enabling evaluation of the system under a greater variety of constraints. Second, the third-order model of the train system developed in this thesis was simulated and its performance under a variety of tests was compared to that of the second-order system.

5.2 Conclusions

On the basis of the results reported in the preceding chapter the following conclusions may be stated.

1. The response of the train system to any perturbation depends to a great extent on the index of performance or cost functional chosen. This is of course to be expected since our only criterion determining the behaviour of the system is the minimization of the cost functional over the time interval during which the control variable is non-zero. We can therefore exert a great deal of control over the regulator performance by altering the weighting factors of the terms within the cost functional.

2. To achieve an optimal controller which will regulate for both headway and position schedule it is necessary to specify a cost functional of the form

$$J = \int_0^{\infty} \left[\sum_{i=1}^N w_i^2(t) + \sum_{i=1}^{N-1} q_i (e_{i1} - e_{i+1,1})^2 + \sum_{i=1}^N q'_i e_{i1}^2 \right] dt \quad (5.1)$$

where $w_i(t)$ is the control variable and q_i and q'_i are non-negative weighting factors.

This statement is true for both the second and third-order models.

3. On the basis of the tests conducted, the third-order model appears to result in a train regulator which better meets the requirement that excessive jerk forces be minimized. More research on the reaction of the third-order system to various types of catastrophic failure would be needed before any conclusions could be made concerning this phase of the study. In the interest of safety and reliability, an overriding control system that would function to apply maximum stopping power to all trains once any train was detected to be decelerating at an abnormally high rate would probably be most satisfactory. Under this condition equation 2.8 would govern the number of collisions.
4. It should be clear from the work of this thesis that it is possible to thoroughly analyze a train regulating system by developing a mathematical model of the string of trains, applying known control theory to develop a controller, and simulating

the resulting system on an analogue computer. Hence it is unnecessary to use trial and error approaches which have characterized the area of train technology in the past.

5.3 Suggestions for Future Research

Possible future research on the subject of control of high speed vehicles falls into two categories -- straightforward extensions of the present work and analysis of completely different aspects of high speed train operation.

Under the first category it would be profitable to re-examine the optimal controllers obtained when the constants d'_i and c'_i in 2.22 are set to values other than unity. Increasing values for d'_i and c'_i would have the effect of reducing the mass and hence inertia of the i th train.

As previously indicated we can "constrain" certain variables in the system only by altering the weighting terms in the cost functionals. It would be useful to determine an optimal regulator using other theory of optimal control (such as non-linear programming or Pontryagin's Maximum Principle) in which constraints such as $|w_i(t)| < 1$ can be considered. This might require a different formulation of the mathematical model (such as discrete differential equations in the case of non-linear programming) but might lead to a more general result.

Under the second category of future research is the problem of the merging of two strings of vehicles. Athans^[9] has reported the following approach. The physical problem is indicated in Figure 5-1.

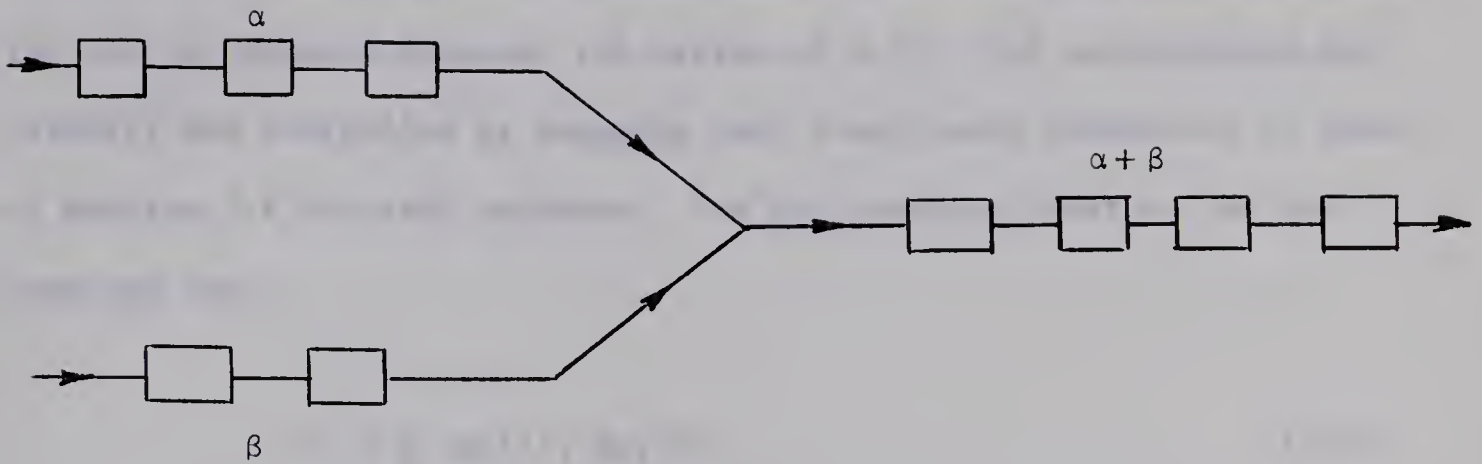


FIGURE 5-1 THE MERGING PROBLEM

The problem can be broken into two parts -- first, what is the best merging order and second, given a merging order what is the best way to control acceleration and velocity? If it is assumed that merging begins far back from the merge point then for a given merging sequence the vehicles can be considered to be on one track. The number of possible merging sequences is given by

$$M = \frac{(\alpha + \beta)!}{\alpha! \beta!} \quad (5.2)$$

So for the γ th possible merging sequence, the motion of the i th vehicle is governed by the differential equation

$$\ddot{x}_{i\gamma}(t) = -a_i \dot{x}_{i\gamma}(t) + b_i u_{i\gamma}(t) \quad (5.3)$$

where $x(t)$ is the position of the i th train and a_i and b_i are the drag and mass coefficients of equation 2.25.

Error state variables can be introduced as before giving for the γ th merging sequence the system of 2.25. The acceleration and velocity are controlled by assuming cost functionals identical to those of section 2.4 for each sequence. For each merging order we can compute the cost

$$J_{\gamma} = \frac{1}{2} \langle e_{\gamma}(t), \hat{K} e_{\gamma}(t) \rangle \quad (5.4)$$

and chose γ^* such that $J_{\gamma^*} < J_{\gamma}$ for all $\gamma \neq \gamma^*$, hence determining the best merging order.

It is thought that since the third-order formulation yielded a better regulator from the standpoint of eliminating excessive jerk, it might be profitable to use this formulation in place of that above and compare merging performance of the two systems.

If the third-order system proved satisfactory in this regard, the optimal control system could then regulate most of the necessary movements of an actual system of trains including slowing down for curves, stations etc. and re-merging with the main guideway.

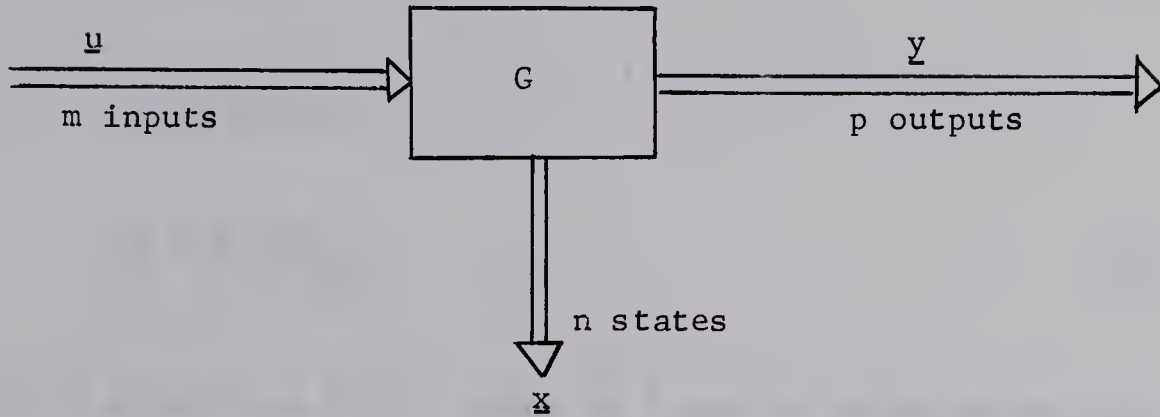
BIBLIOGRAPHY

1. Michaels, E.L., "Today's Need For Balanced Urban Transit Systems",
IEEE Spectrum, vol. 4, pp. 87-91, December 1967.
2. Fujii, M., "New Tokaido Line", Proc. IEEE, vol. 56, pp. 625-645,
April 1968.
3. Levine, W.S. and M. Athans, "On the Optimal Error Regulation of a
String of Moving Vehicles", IEEE Trans. on Automatic
Control, vol. AC-11, pp. 355-361, July 1966.
4. Levine, W.S. and M. Athans, "On the Linear Optimal Control of a
String of Moving Vehicles", S.M. Thesis, Mass. Inst.
Tech., Cambridge, Mass., June 1965.
5. Friedlander, G.D., "Railway vs Highway -- the Zoom of Things of
Come", IEEE Spectrum, vol. 4, pp. 62-76, September 1967.
6. Hajdu, L.P., K.W. Gardiner, H. Tamura and G.L. Pressman, "Design and
Control Considerations for Automated Ground Transportation
Systems", Proc. IEEE, vol. 56, pp. 493-513, April 1968.
7. Athans, M. and P.L. Falb, "Optimal Control", McGraw Hill, New York,
1966, Chapter 9.
8. Kleinman, D.L., "On the Linear Regulator Problem and the Matrix
Riccati Equation", Report ESL-R-271, Mass. Inst. Tech.,
Cambridge, Mass., 1966.

9. Athans, M., "Application of Optimal Control to High Speed Ground Transportation Problems", Paper presented to the 6th Allerton Conference on Circuit and System Theory, Monticello, Ill., October 1968.

APPENDIX 1

THE MEANING OF STATE-CONTROLLABILITY



A system G defined by the state equation

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) \quad (\text{A1.1})$$

is said to be state-controllable if knowing the matrices $\underline{A}$ and $\underline{B}$ it is possible to construct an input $\underline{u}$ which will take the state $\underline{x}(t_0)$ to $\underline{x}(t_f) = \underline{0}$ in some finite time interval T.

It is shown in [7] that the initial state can be written as

$$\underline{x}(t_0) = - \begin{bmatrix} \underline{B} & \underline{A}\underline{B} & \underline{A}^2\underline{B} & \dots & \underline{A}^{n-1}\underline{B} \end{bmatrix} \begin{bmatrix} \underline{v}_0(t_0, t_f) \\ \vdots \\ \underline{v}_{n-1}(t_0, t_f) \end{bmatrix} \quad (\text{A1.2})$$

or

$$\underline{x}(t_0) = \underline{S}\underline{V} \quad (\text{A1.3})$$

where

$$\underline{v}_k(t_0, \tau) = - \int_{t_0}^{t_f} \alpha_k(t_0 - \tau) \underline{u}(\tau) d\tau \quad (\text{A1.4})$$

From A1.3,

$$\underline{v} = \underline{S}^{-1} \underline{x}(t_0) \quad (\text{A1.5})$$

which by A1.4 stipulates $\underline{u}(t)$. Hence A1.5 must be valid which means that $\underline{S}^{-1}$ must exist (i.e. $\det \underline{S} \neq 0$). This requires that $\underline{S}$ (dimensions $n \times n$) be of rank n for our system to exhibit state controllability. [n is the dimension of $\underline{x}(t)$ and m the dimension of $\underline{u}(t)$]

Cost	J ₁	J ₂	J ₃	J ₄	J ₅	J ₆	J ₇	J ₈	J ₉
k ₂₁	0.789	1.764	2.494	3.527	1.764	2.494	0.0	2.494	4.344
k ₂₂	0.552	0.996	1.263	1.587	1.764	2.494	1.920	2.968	2.095
k ₂₃	-0.577	-1.291	-1.826	-2.582	-1.291	-1.826	0.0	-1.826	-1.054
k ₂₄	-0.371	-0.652	-0.819	-1.021	-1.291	-1.826	-1.523	-0.456	-0.329
k ₂₅	-0.211	-0.473	-0.668	-0.945	-0.473	-0.669	0.0	-0.668	-0.128
k ₂₆	-0.181	-0.343	-0.444	-0.567	-0.473	-0.669	-0.397	-0.195	-0.059
k ₄₁	-0.577	-1.291	-1.826	-2.582	-1.291	-1.826	0.0	-1.826	-1.054
k ₄₂	-0.371	-0.652	-0.819	-1.021	-1.291	-1.826	-1.523	-0.456	-0.329
k ₄₃	1.155	2.582	3.652	5.164	2.582	3.651	0.0	3.652	5.271
k ₄₄	0.742	1.305	1.639	2.040	2.582	3.651	3.045	3.229	2.365
k ₄₅	-0.577	-1.291	-1.826	-2.582	-1.291	-1.826	0.0	-1.826	-1.054
k ₄₆	-0.371	-0.652	-0.819	-1.021	-1.291	-1.826	-1.523	-0.456	-0.329
k ₆₁	-0.211	-0.473	-0.668	-0.945	-0.473	-0.669	0.0	-0.668	-0.128
k ₆₂	-0.181	-0.343	-0.444	-0.567	-0.473	-0.669	-0.397	-0.195	-0.059
k ₆₃	-0.577	-1.291	-1.826	-2.582	-1.291	-1.826	0.0	-1.826	-1.054
k ₆₄	-0.371	-0.652	-0.819	-1.021	-1.291	-1.826	-1.523	-0.456	-0.329
k ₆₅	0.789	1.764	2.494	3.527	-1.764	2.494	0.0	2.494	4.344
k ₆₆	0.552	0.996	1.263	1.586	-1.764	2.494	1.920	2.968	2.095

TABLE A2-1 DIGITAL COMPUTER RESULTS FOR 2ND-ORDER SYSTEM

Cost	J ₁	J ₂	J ₃	J ₄	J ₅	J ₆	J ₇	J ₈	J ₉
k ₃₁	0.789	1.764	2.494	3.527	1.764	2.494	0.0	2.494	4.344
k ₃₂	1.579	2.635	3.282	4.166	3.284	4.300	2.494	5.109	5.448
k ₃₃	0.927	1.318	1.515	1.803	1.521	1.806	1.263	2.318	2.440
k ₃₄	-0.577	-1.291	-1.826	-2.582	-1.291	-1.826	0.0	-1.826	-1.054
k ₃₅	-1.011	-1.696	-2.137	-2.622	-2.219	-2.932	-1.826	-1.379	-0.845
k ₃₆	-0.552	-0.781	-0.914	-1.001	-0.928	-1.107	-0.819	-0.414	-0.241
k ₃₇	-0.211	-0.473	-0.668	-0.945	-0.473	-0.669	0.0	-0.668	-0.128
k ₃₈	-0.566	-0.938	-1.183	-1.421	-1.066	-1.377	-0.668	-0.568	-0.139
k ₃₉	-0.373	-0.536	-0.636	-0.686	-0.594	-0.709	-0.443	-0.197	-0.049
k ₆₁	-0.577	-1.291	-1.826	-2.582	-1.291	-1.826	0.0	-1.826	-1.054
k ₆₂	-1.011	-1.696	-2.137	-2.622	-2.219	-2.932	-1.826	-1.379	-0.845
k ₆₃	-0.552	-0.781	-0.914	-1.001	-0.928	-1.107	-0.819	-0.414	-0.241
k ₆₄	1.155	2.582	3.652	5.164	2.582	3.651	0.0	3.652	5.271
k ₆₅	2.024	3.393	4.236	5.368	4.437	5.855	3.652	5.919	6.154
k ₆₆	1.107	1.563	1.794	2.119	1.855	2.203	1.639	2.535	2.632
k ₆₇	-0.577	-1.291	-1.826	-2.582	-1.291	-1.826	0.0	-1.826	-1.054
k ₆₈	-1.011	-1.696	-2.137	-2.622	-2.219	-2.932	-1.826	-1.379	-0.845
k ₆₉	-0.552	-0.781	-0.914	-1.001	-0.928	-1.107	-0.819	-0.414	-0.241
k ₉₁	-0.211	-0.473	-0.668	-0.945	-0.473	-0.669	0.0	-0.668	-0.128
k ₉₂	-0.566	-0.938	-1.183	-1.420	-1.066	-1.377	-0.668	-0.568	-0.139
k ₉₃	-0.373	-0.536	-0.636	-0.686	-0.594	-0.709	-0.443	-0.197	-0.049
k ₉₄	-0.577	-1.291	-1.826	-2.582	-1.291	-1.826	0.0	-1.826	-1.054
k ₉₅	-1.011	-1.696	-2.137	-2.622	-2.219	-2.932	-1.826	-1.379	-0.845
k ₉₆	-0.552	-0.781	-0.914	-1.001	-0.928	-1.107	-0.819	-0.414	-0.241
k ₉₇	0.789	1.764	2.494	3.527	1.763	2.494	0.0	2.494	4.344
k ₉₈	1.579	2.635	3.282	4.166	3.284	4.300	2.494	5.109	5.448
k ₉₉	0.927	1.318	1.515	1.803	1.521	1.806	1.263	2.318	2.440

TABLE A2-2 DIGITAL COMPUTER RESULTS FOR 3RD-ORDER SYSTEM

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